# Radiation from charges in the continuum limit

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# Abstract

It is known that an accelerating charge radiates according to Larmor formula. On the other hand, any DC current following a curvilinear path, consists of accelerating charges, but in such case the radiated power is 0. The scope of this paper is to analyze and quantify how a system of charges goes from a radiating state to a non radiating state when the charges distribution goes to the continuum limit. Understanding this is important from the theoretical point of view and the results of this work are applicable to particle accelerator, cyclotron and other high energy devices.

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## I. INTRODUCTION

There are many physical configurations in which discrete charges are in acceleration, but in spite of that they behave almost like steady state DC. This situation is encountered in any DC or low frequency electrical circuit, because in spite of the fact that the charge distribution is always discrete and each charge accelerates in the influence of the electric field, one may consider the charge distribution as almost continuous. For example the model used for conducting materials is of an average drift velocity for positive and negative charges which is proportional to the electric field  $\mathbf{v}^{\pm} = \pm \mu^{\pm} \mathbf{E}$ ,  $\mu^{\pm}$  being the mobility of the positive and negative charges, respectively. Hence the current density is  $\mathbf{J} = \rho^+ \mathbf{v}^+ + \rho^- \mathbf{v}^- =$  $(\rho^+ \mu^+ - \rho^- \mu^-)\mathbf{E}$ , where  $\rho^{\pm}$  is the charge density of the positive and negative charge carriers,  $\rho^-$  being negative, so that the total charge density is  $\rho = \rho^+ + \rho^- = \epsilon_0 \nabla \cdot \mathbf{E}$ . In case the charges are protons and electrons, in a conductor, the protons mobility is  $\mu^+ = 0$  and  $\rho^+ = -\rho^-$ , so that usually  $\nabla \cdot \mathbf{E} = 0$  like in free space. In addition, the model considers  $\rho^$ to be almost uniform, so that one defines the conductivity  $\sigma = -\rho^- \mu^-$ , resulting in Ohm's law  $\mathbf{J} = \sigma \mathbf{E}$ .

Another situation, for which charges are somehow "more discrete", are DC or low frequency ion drift devices, and for those we usually have one type of charge carriers, say positive ions. Here we use  $\mathbf{J} = \rho \mathbf{v} = \rho \mu \mathbf{E}$ , and one may not assume the charge density is uniform, but rather has to use Gauss's law  $\epsilon_0 \nabla \cdot \mathbf{E} = \rho$ , resulting in a set of nonlinear equations. In principle, such problem is time dependent, because discrete ions move in the space, but it appears that the DC approximation  $\nabla \cdot \mathbf{J} = 0$  works very well for such cases [13–16].

In either of the above situations, currents may follow curvilinear paths, in which case, the charges clearly accelerate, also if the magnitude of their velocity is constant, and still the DC model works well for most practical cases for which the density of the charge carriers is big.

The purpose of this work is to analyze and quantify how a system of charges goes from a radiating state to a non radiating state when the charges distribution goes to the continuum limit. Certainly, the radiation has to disappear gradually, so that this vanishing may be quantified. Some preliminary work [12] has been done in this direction, but because this work has been done a priori in a non relativistic approach, its results are inaccurate and incomplete.

Understanding the physics of the transition from a radiative state to a non radiative state is important from the theoretical point of view, but not only - this work analyzes radiation from rotating charges and its results are applicable to particle accelerator, cyclotron and other high energy problems [17–22].

It should be mentioned that the continuum limit of a charge distribution does not exist in reality, because the electric charge is quantized as integer multiples of the electron charge. However, this physical fact does not apply any constraint on the classical (non quantum) electrodynamics, which handles in a consistent way continuum distribution charges and DC. This work being carried out in the classical electrodynamics framework, there is nothing to prevent us from considering the continuum distribution limit.

The calculation is done in a canonical configuration of charges in circular motion at constant speed. The configuration and the formulation are explained in section 2.

In section 3 we calculate the fields, explain their behavior and derive an exact expression for the radiated power.

In section 4 we calculate the radiation reaction and show that the power needed to support the radiation equals the radiated power.

In section 5 we derive an asymptotic result for the case the number of charges is big (continuum limit). As a special case, we derive in section 6 the limit for slow charges, and this case is of importance because it represents the typical situation of DC currents in devices as discussed before. The mathematical derivations in those sections are tedious, but the results are simple and conclusive.

The paper is ended with some concluding remarks.

The work is written in SI units, and we shall use the known constants: vacuum permittivity  $\epsilon_0 = 8.85 \times 10^{-12}$  F/m, vacuum permeability  $\mu_0 = 4\pi \times 10^{-7}$  H/m, free space impedance  $\eta_0 = \sqrt{\mu_0/\epsilon_0} = 376.73 \Omega$  and speed of light in vacuum  $c = 1/\sqrt{\mu_0\epsilon_0} \approx 3 \times 10^8$  m/sec.

## **II. CONFIGURATION AND FORMULATION**

A total amount of charge q is rotating in a circle of radius d at constant speed v so that the angular velocity is  $\omega = v/d$ . The charge q is "split" into N charges of value q/N, uniformly distributed around the circle, so that the charge number k is at the angle  $\omega t + 2\pi k/N$ , where



FIG. 1: (color online) N charges (here N = 3) of magnitude q/N each, in circular motion on the xy plane at radius d around the z axis. The far observer's spherical coordinates are also shown in the figure.

k = 0, 1, ..N - 1.

The configuration is shown in Figure 1, for N = 3.

The location of the charge k as function of time is given by:

$$\mathbf{r}'_{k}(t) = d[\widehat{\mathbf{x}}\cos(\omega t + 2\pi k/N) + \widehat{\mathbf{y}}\sin(\omega t + 2\pi k/N)]$$
(1)

The fields propagate with the speed of light c. Hence the fields at the observer location **r** at time t are influenced by the motion of each charge, at an *earlier* (retarded) time. Specifically, the fields are influenced by the motion of the charge k at time  $t'_k$  so that

$$R_k \equiv |\mathbf{r} - \mathbf{r}'_k(t'_k)| = c(t - t'_k) \tag{2}$$

At large distance from the charges, one may approximate:

$$R_k \approx r - d\sin\theta\cos\phi_k \tag{3}$$



FIG. 2: (color online) The dark colored spheres represent the charges at the current position at time t and the light colored spheres represent the charges at the retarded positions at times  $t'_k$ . The retarded positions are connected with dashed lines to the observer location, and those distances are called  $R_0$ ,  $R_1$  and  $R_2$ . This emphasizes that the field at observer is determined by the velocities and accelerations of the charges at the retarded times.

where

$$\phi_k \equiv \omega t'_k + 2\pi k/N - \varphi, \tag{4}$$

hence the retarded time  $t'_k$  may be calculated from the following implicit equation:

$$t'_{k} = t - r/c + (d/c)\sin\theta\cos\phi_{k},\tag{5}$$

which may be solved numerically by setting a "1st guess"  $t'_k = t - r/c$  in the right side of eq. (5) and recalculate  $t'_k$  until convergence is obtained.

Figure 2 emphasizes the meaning of retarded positions of the charges.

Our purpose is to calculate the power radiated by those rotating charges. Of course, for N = 1 the result is known by Larmor formula (and will be confirmed later on).

The general form of the Larmor formula [1–11] is:

$$P_{\text{Larmor}} = \frac{q^2 \gamma^6}{6\pi \epsilon_0 c} (\dot{\beta}^2 - (\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2), \tag{6}$$

where

$$\boldsymbol{\beta} \equiv \mathbf{v}/c. \tag{7}$$

and

$$\gamma \equiv 1/\sqrt{1-\beta^2} \tag{8}$$

If one defines the angle between the velocity and the acceleration as  $\alpha$ , one may express  $(\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}})^2 = (\beta \dot{\beta} \sin \alpha)^2$  an rewrite

$$P_{\text{Larmor}} = \frac{q^2 \gamma^6}{6\pi\epsilon_0 c} \dot{\beta}^2 (1 - \beta^2 \sin^2 \alpha).$$
(9)

In our case of circular motion, the velocity is perpendicular to the acceleration, so that  $\sin^2 \alpha = 1$ . Therefore by replacing  $1 - \beta^2 = 1/\gamma^2$ , the Larmor formula simplifies for our case to

$$P|_{N=1} = \frac{q^2 \gamma^4 \dot{\beta}^2}{6\pi\epsilon_0 c} = \frac{q^2 \gamma^4 a^2}{6\pi\epsilon_0 c^3},$$
(10)

where a is the acceleration. We will calculate how this power decreases when the number of charges N increases.

# **III. FIELDS AND POWER CALCULATION**

To calculate the power radiated from the collection of charges in Figure 1, one needs only the far fields, i.e. those who behave like 1/R. The far electric and magnetic fields due to the moving charge k are given by [2, 5]:

$$\mathbf{E}_{k} = (q/N)\mu_{0} \frac{\widehat{\mathbf{R}}_{k} \times [(\widehat{\mathbf{R}}_{k} - \boldsymbol{\beta}_{k}) \times \mathbf{a}_{k}]}{4\pi (1 - \boldsymbol{\beta}_{k} \cdot \widehat{\mathbf{R}}_{k})^{3} R_{k}}$$
(11)

and

$$\mathbf{H}_{k} = \widehat{\mathbf{R}}_{k} \times \mathbf{E}_{k} / \eta_{0} \tag{12}$$

where  $\hat{\mathbf{R}}_k$  is the unit vector pointing from the position of the charge to observer and  $R_k$ is the distance between the charge and the observer, as defined in eq. (2) - see Figure 2.  $\boldsymbol{\beta}_k = \mathbf{v}_k/c = \dot{\mathbf{r}}'_k/c$  is the velocity relative to c and  $\mathbf{a}_k = \dot{\mathbf{v}}_k$  is the acceleration of the charge. All the dynamical variables are evaluated at the retarded time (defined in eq. (5)).

Defining  $\hat{\mathbf{r}}$  as the unit vector pointing from the coordinates origin to the observer, we may calculate in the far field the difference:

$$\widehat{\mathbf{R}}_{k} - \widehat{\mathbf{r}} = \frac{\mathbf{r} - \mathbf{r}_{k}'}{|\mathbf{r} - \mathbf{r}_{k}'|} - \frac{\mathbf{r}}{r} \approx -\frac{\mathbf{r}_{k}'}{r}$$
(13)

Hence one may use  $\hat{\mathbf{r}}$  instead of  $\hat{\mathbf{R}}_k$  in eqs. (11) and (12) with an error of order  $1/R^2$ , which does not affect the calculations of the radiated power. Also, in the denominator of eq. (11) we may set  $R_k = r$ , as always done for far field. So we express the electric and magnetic fields as the sum of the contribution from all the charges:

$$\mathbf{E} = \frac{\mu_0 q}{4\pi r N} \sum_{k=0}^{N-1} \frac{\widehat{\mathbf{r}} \times [(\widehat{\mathbf{r}} - \boldsymbol{\beta}_k) \times \mathbf{a}_k]}{(1 - \boldsymbol{\beta}_k \cdot \widehat{\mathbf{r}})^3}$$
(14)

and

$$\mathbf{H} = \widehat{\mathbf{r}} \times \mathbf{E} / \eta_0 \tag{15}$$

Now using eq. (1), we evaluate  $\hat{\mathbf{r}} \times [(\hat{\mathbf{r}} - \boldsymbol{\beta}_k) \times \mathbf{a}_k]$  and express it in spherical coordinates:

$$\widehat{\mathbf{r}} \times [(\widehat{\mathbf{r}} - \boldsymbol{\beta}_k) \times \mathbf{a}_k = a[\widehat{\boldsymbol{\theta}}\cos\theta\cos\phi_k + \widehat{\boldsymbol{\varphi}}(\beta\sin\theta + \sin\phi_k)]$$
(16)

and  $\boldsymbol{\beta}_k \cdot \hat{\mathbf{r}}$  evaluates to

$$\boldsymbol{\beta}_k \cdot \hat{\mathbf{r}} = -\beta \sin \theta \sin \phi_k. \tag{17}$$

Setting those results into eq. (14), we obtain

$$\mathbf{E} = \frac{\mu_0 q a}{4\pi r N} [\widehat{\boldsymbol{\theta}} \cos \theta \operatorname{Fc} + \widehat{\boldsymbol{\varphi}} \operatorname{Fs}]$$
(18)

and

$$\mathbf{H} = \frac{qa}{4\pi rcN} [-\widehat{\boldsymbol{\theta}} \operatorname{Fs} + \widehat{\boldsymbol{\varphi}} \cos \theta \operatorname{Fc}]$$
(19)

where the functions Fc and Fs are defined as:

$$Fc(t,\varphi,\theta,\beta,N) \equiv \sum_{k=0}^{N-1} fc(\phi_k)$$
(20)

and

$$Fs(t,\varphi,\theta,\beta,N) \equiv \sum_{k=0}^{N-1} fs(\phi_k)$$
(21)

and the functions fc and fs are defined as

$$fc(\phi_k) \equiv \frac{\cos \phi_k}{(1+p\sin \phi_k)^3},\tag{22}$$

and

$$fs(\phi_k) \equiv \frac{p + \sin \phi_k}{(1 + p \sin \phi_k)^3},\tag{23}$$

and p is defined as

$$p \equiv \beta \sin \theta \tag{24}$$

and is a parameter which controls the behavior of  $\phi_k$ , as will be soon shown.

The power per unit of normal area (or Poynting vector) is given by,  $\mathbf{S} = \mathbf{E} \times \mathbf{H} = \hat{\mathbf{r}} E^2 / \eta_0$ which results in

$$\mathbf{S} = \widehat{\mathbf{r}} \left[ \frac{qa}{4\pi cNr} \right]^2 \eta_0 \left| \widehat{\boldsymbol{\varphi}} \operatorname{Fs} + \widehat{\boldsymbol{\theta}} \cos \theta \operatorname{Fc} \right|^2$$
(25)

The total power is calculated via

$$P = r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta \, \mathbf{S} \cdot \hat{\mathbf{r}}$$
(26)

which results in

$$P = P|_{N=1} G(t,\beta,N) = \frac{q^2 a^2 \gamma^4}{6\pi\epsilon_0 c^3} G(t,\beta,N).$$
(27)

Here we factored out the Larmor formula for the radiation of a single charge - see eq. (10), so that  $G(t, \beta, N)$  is dimensionless and represents the decay of the power. The function  $G(t, \beta, N)$  is given by

$$G(t,\beta,N) \equiv \frac{3}{8\pi\gamma^4 N^2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \ F(t,\varphi,\theta,\beta,N)$$
(28)

hence G = 1 for N = 1, for any t or  $\beta$  and the function F is

$$F(t,\varphi,\theta,\beta,N) \equiv \left|\widehat{\varphi}\operatorname{Fs} + \widehat{\theta}\cos\theta\operatorname{Fc}\right|^2 = \operatorname{Fs}^2 + \cos^2\theta\operatorname{Fc}^2$$
(29)

Now we eliminate  $t'_k$  from eq. (5) and rewrite the implicit eqs. (5) and (4) in terms of  $\phi_k$ 

$$\phi_k = \omega(t - r/c) - \varphi + 2\pi k/N + p\cos\phi_k \tag{30}$$

This allows us to change variable  $\varphi' = \varphi - \omega(t - r/c)$  in (28) obtaining:

$$G(t,\beta,N) = \frac{3}{8\pi\gamma^4 N^2} \int_{-\omega(t-r/c)}^{2\pi-\omega(t-r/c)} d\varphi' \int_0^{\pi} d\theta \sin\theta \,(\mathrm{Fs}^2 + \cos^2\theta \,\mathrm{Fc}^2),\tag{31}$$

so that eq. (30) is rewritten as

$$\phi_k = -\varphi' + 2\pi k/N + p \,\cos\phi_k,\tag{32}$$

We see that if  $\phi_k$  and  $\varphi'$  satisfy eq. (32), also  $\phi_k - 2\pi$  and  $\varphi' + 2\pi$  satisfy it, hence  $\cos \phi_k$ and  $\sin \phi_k$  are periodic functions of  $\varphi'$ , with a periodicity of  $2\pi$ . Therefore, the  $d\varphi'$  integral in eq. (31) may be evaluated over any period of  $2\pi$ , showing that G (and therefore also the radiated power P) does not depend on time, so that we may simplify eq. (31) to

$$G(\beta, N) = \frac{3}{8\pi\gamma^4 N^2} \int_0^{2\pi} d\varphi' \int_0^{\pi} d\theta \sin\theta \,(\mathrm{Fs}^2 + \cos^2\theta \,\mathrm{Fc}^2). \tag{33}$$

We redefined G to be time independent, so that fc, fs, Fc and Fs in eqs. (22), (23), (20) and (21) become functions of  $\varphi'$  instead of t and  $\varphi$ .

For understanding the behavior of Fs and Fc, we plot fc, fs in eqs. (22) and (23), and

theirs sums (eqs. (20) and (21)) for different parameters.

Clearly for very small p in eq. (32),  $\phi_k \approx -\varphi' + 2\pi k/N$ , hence the cosine or sine of  $\phi_k$ equal approximately to the cosine or sine of  $-\varphi' + 2\pi k/N$ , so that both have harmonic shapes as function of  $\varphi'$ . In such case fc  $\approx \cos(-\varphi' + 2\pi k/N)$  and fs  $\approx \sin(-\varphi' + 2\pi k/N)$ , hence they sum to a small value as observed in Figures 3 and 4. For N = 3 (Figure 3) the amplitudes of Fc and Fs are around 0.1, and this decreases with N, as may be seen in Figure 4, for N = 10.



FIG. 3: (color online) The fc in eq. (22) and their sum is shown in panel (a) and the fs in eq. (23) and their sum is shown in panel (b) for N = 3 and p = 0.1.



FIG. 4: (color online) The fc in eq. (22) and their sum is shown in panel (a) and the fs in eq. (23) and their sum is shown in panel (b) for N = 10 and p = 0.1.

As p increases, fc and fs in eqs. (22) and (23) get more distorted, hence they sum to bigger amplitudes, as observed in Figures 5, 6, 7 and 8.

In Figure 5, showing the behavior for p = 0.3 and N = 3, Fc and Fs have amplitudes of

0.87 and 0.76, respectively, and those decrease for N = 10 (Figure 6) to 0.0086 and 0.0082. Also we see that for p = 0.3, although fc and fs are distorted, the sums Fc and Fs are almost undistorted, unlike for the p = 0.5 and N = 3 case in Figure 7.



FIG. 5: (color online) The fc in eq. (22) and their sum is shown in panel (a) and the fs in eq. (23) and their sum is shown in panel (b) for N = 3 and p = 0.3.



FIG. 6: (color online) The fc in eq. (22) and their sum is shown in panel (a) and the fs in eq. (23) and their sum is shown in panel (b) for N = 10 and p = 0.3.

The case of p = 0.5 is shown in Figures 7 and 8. In this case, not only fc and fs are distorted, but also the sums Fc and Fs, however the distortion of the sum decreases when the number of charges N increases, as may be seen for the case N = 10 in Figure 8. In the last case the amplitudes of Fc and Fs are 0.59 and 0.5, much bigger than in the parallel case with p = 0.3.

To summarize, the functions Fc and Fs increase with p and decrease with N, tending to undistorted harmonic functions, for big values of N.



FIG. 7: (color online) The fc in eq. (22) and their sum is shown in panel (a) and the fs in eq. (23) and their sum is shown in panel (b) for N = 3 and p = 0.5.



FIG. 8: (color online) The fc in eq. (22) and their sum is shown in panel (a) and the fs in eq. (23) and their sum is shown in panel (b) for N = 10 and p = 0.5.

It is interesting to remark that the average of the functions fc and fs is always 0, although this is not always visible for the fs functions. This may be proved by calculating

$$\langle \text{fc,s} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \text{fc,s}(\phi_k) = \frac{1}{2\pi} \int_0^{2\pi} d\varphi' \frac{\text{hc,s}(\phi_k)}{(1+p\sin\phi_k)^3}$$
(34)

where for brevity we called fc,s the functions fc or fs, and we called their average  $\langle fc,s \rangle$ . We also use the abbreviation hc,s for the functions hc and hs defined as

$$hc(\phi_k) \equiv \cos \phi_k \tag{35}$$

$$hs(\phi_k) \equiv p + \sin \phi_k \tag{36}$$

for the fc and fs average calculation, respectively. We change variable from  $\varphi'$  to  $\phi_k$ , and we find from eq. (32) that

$$d\varphi'/d\phi_k = -1 - p \,\sin\phi_k,\tag{37}$$

getting:

$$\langle \text{fc,s} \rangle = -\frac{1}{2\pi} \int_{\phi_k(0)}^{\phi_k(0)-2\pi} d\phi_k \frac{\text{hc,s}(\phi_k)}{(1+p\sin\phi_k)^2},$$
 (38)

where  $\phi_k(0)$  is the value of  $\phi_k$  at  $\varphi' = 0$ . Because the integrand has a periodicity of  $2\pi$ , one may integrate over any period of  $2\pi$ , obtaining:

$$\langle \mathrm{fc,s} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi_k \frac{\mathrm{hc,s}(\phi_k)}{(1+p\sin\phi_k)^2}.$$
(39)

This integral is solved by the residue method on the complex plane, in Appendix A, and its result is 0.

Now we continue with the calculation of G in eq. (33). Rewriting eq. (32) for the charge m instead of k results in

$$\phi_m = -\varphi' + 2\pi m/N + p \,\cos\phi_m,\tag{40}$$

which may be rewritten as

$$\phi_m = -(\varphi' - 2\pi (m-k)/N) + 2\pi k/N + p \, \cos \phi_m, \tag{41}$$

showing that if we could explicitly express  $\phi_k(\varphi')$  from eq. (32),  $\phi_m$  would be the same function of  $\varphi'$  only shifted:

$$\phi_k(\varphi') = \phi_m(\varphi' - 2\pi(m-k)/N) \tag{42}$$

as also evident from Figures 3 - 8. Therefore both Fs and Fc functions remain unchanged for a shift of multiples of  $2\pi/N$ , say:

$$\operatorname{Fc}(\varphi' + 2\pi/N), \theta, \beta, N) = \sum_{k=0}^{N-1} \frac{\cos \phi_k (\varphi' + 2\pi/N)}{(1 + p \sin \phi_k (\varphi' + 2\pi/N))^3} = \sum_{k=0}^{N-1} \frac{\cos \phi_{k+1}(\varphi')}{(1 + p \sin \phi_{k+1}(\varphi'))^3}.$$
 (43)

and the sum being on all charges (and k is modulo N), we are left with the same result. We may therefore rewrite in eq. (33)

$$\int_{0}^{2\pi} d\varphi' = \sum_{n=0}^{N-1} \int_{2\pi n/N}^{2\pi (n+1)/N} d\varphi'$$
(44)

and after changing variable  $\varphi'' = \varphi' - 2\pi n/N$ , we are left with N identical integrals over the period 0 to  $2\pi/N$ . For simplicity we rename  $\varphi''$  back to  $\varphi'$  and rewrite the function G as:

$$G(\beta, N) = \frac{3}{8\pi\gamma^4 N} \int_0^{2\pi/N} d\varphi' \int_0^{\pi} d\theta \sin\theta \,(\mathrm{Fs}^2 + \cos^2\theta \,\mathrm{Fc}^2),\tag{45}$$

and this will significantly reduce the time of a numerical integration. Now looking at the  $\theta$  dependence of Fs and Fc, we remark from eq. (32) that  $\phi_k$  depends on  $p = \beta \sin \theta$ , hence it is invariant under replacing  $\theta$  by  $\pi - \theta$ , and so is  $\cos^2 \theta$ . We may therefore replace the integration from 0 to  $\pi$  by twice the integration from 0 to  $\pi/2$ , getting

$$G(\beta, N) = \frac{3}{4\pi\gamma^4 N} \int_0^{2\pi/N} d\varphi' \int_0^{\pi/2} d\theta \sin\theta \,(\mathrm{Fs}^2 + \cos^2\theta \,\mathrm{Fc}^2),\tag{46}$$

Now we change to the variable p defined in eq. (24) and rewrite eq. (46) obtaining:

$$G(\beta, N) = \frac{3}{4\pi N \gamma^4 \beta^2} \int_0^{2\pi/N} d\varphi' \int_0^\beta dp \, p \left( \mathrm{Fs}^2 / \sqrt{1 - (p/\beta)^2} + \mathrm{Fc}^2 \sqrt{1 - (p/\beta)^2} \right), \quad (47)$$

For the case of N = 1 we know G must be 1, but for consistency we shall prove it:

$$G(\beta, 1) = \frac{3}{4\pi\gamma^4\beta^2} \int_0^{2\pi} d\varphi' \int_0^\beta dp \, p \left( \left[ \frac{p + \sin\phi_0}{(1 + p\sin\phi_0)^3} \right]^2 / \sqrt{1 - (p/\beta)^2} + \left[ \frac{\cos\phi_0}{(1 + p\sin\phi_0)^3} \right]^2 \sqrt{1 - (p/\beta)^2} \right), \tag{48}$$

where Fc<sup>2</sup> and Fs<sup>2</sup> reduced here to a single term. By changing integration order, we may perform the  $d\varphi'$  integration by the change of variable  $d\varphi'/d\phi_0$  defined in eq. (37), obtaining

$$G(\beta,1) = \frac{3}{4\pi\gamma^4\beta^2} \int_0^\beta dp \, p \int_{\phi_0(0)-2\pi}^{\phi_0(0)} d\phi_0 \left( \frac{(p+\sin\phi_0)^2}{(1+p\sin\phi_0)^5\sqrt{1-(p/\beta)^2}} + \frac{\cos^2\phi_0\sqrt{1-(p/\beta)^2}}{(1+p\sin\phi_0)^5} \right)$$
(49)

All the functions have a  $2\pi$  periodicity on  $\phi_0$ , so one may use any limits of interval  $2\pi$  for  $\phi_0$ . By changing variable  $z = \exp(i\phi_0)$  and using the residue method on the complex plane we obtain

$$\int_{0}^{2\pi} d\phi_0 \frac{\cos^2 \phi_0}{(1+p\sin\phi_0)^5} = \frac{\pi}{4} \frac{4+p^2}{(1-p^2)^{7/2}}$$
(50)

and

$$\int_{0}^{2\pi} d\phi_0 \frac{(p+\sin\phi_0)^2}{(1+p\sin\phi_0)^5} = \frac{\pi}{4} \frac{4+3p^2}{(1-p^2)^{5/2}},\tag{51}$$

hence

$$G(\beta,1) = \frac{3}{16\gamma^4\beta^2} \int_0^\beta dp \, p \left[ \frac{(4+p^2)\sqrt{1-(p/\beta)^2}}{(1-p^2)^{7/2}} + \frac{(4+3p^2)}{(1-p^2)^{5/2}\sqrt{1-(p/\beta)^2}}, \right]$$
(52)

which results in

$$G(\beta,1) = \frac{3}{16\gamma^4\beta^2} \left[ \frac{2\beta^2(2-\beta^2)}{3(1-\beta^2)^2} + \frac{2\beta^2(6+\beta^2)}{3(1-\beta^2)^2} \right] = \frac{3}{16\gamma^4\beta^2} \frac{16\beta^2}{3(1-\beta^2)^2} = 1$$
(53)

We perform now the calculation in eq. (47) numerically. Knowing that  $G(\beta, 1) = 1$ , this calculation actually shows the power radiated by N charges, divided by the power radiated by a single charge. The results are shown as function of N for different values of  $\beta$  in Figure 9.

The radiated power calculated above, must be supplied by a power source which adds the necessary energy to overcome the radiation reaction. This radiation reaction is calculated



FIG. 9: Result of G (eq. (47)) as function of the number of charges N, for different values of  $\beta$ . For big N, G goes asymptotically to 0 and as smaller  $\beta$  is, G goes faster to 0.

in the following section.

# IV. THE RADIATION REACTION

We calculate now the Lorentz force on the charges and the resulting radiation resistance power for comparing with the radiated power. We will need the electric field on charge n at time t due to charge m at its retarded position at the earlier time t', so that:

$$|\mathbf{r}'_{n}(t) - \mathbf{r}'_{m}(t')| = c(t - t').$$
(54)

Using the expression for  $\mathbf{r'}_k$  in eq. (1) we obtain

$$4d^{2}\sin^{2}\left[\frac{\omega(t-t')+2\pi(n-m)/N}{2}\right] = c^{2}(t-t')^{2},$$
(55)

which is exact for any m and n. By definition, t - t' > 0, so to take the correct square root from the left side of this equation, we need to know the connection between m and n. To simplify, we restrict:

$$0 \le m < n \le N - 1,\tag{56}$$

for which we obtain

$$2d\sin\Phi_{nm} = c(t-t'),\tag{57}$$

where  $\Phi_{nm}$  is defined by

$$\Phi_{nm} \equiv \frac{\omega(t-t') + 2\pi(n-m)/N}{2}.$$
(58)

Now we isolate t - t' from eq. (58) and set it in eq. (57), obtaining

$$\Phi_{nm} = \pi (n-m)/N + \beta \sin \Phi_{nm}, \tag{59}$$

which is an implicit equation, that can be solved by setting a 1st guess  $\Phi_{nm} = \pi (n-m)/N$ in the right side of the equation and recalculate  $\Phi_{nm}$  till convergence is obtained.

Figure (10) gives the geometrical interpretation of eqs. (57)-(59).

Let us look at a simple example of solution for eq. (59). Say there are 4 charges, so their locations at t = 0 are: 0°, 90°, 180° and 270° for charges 0, 1, 2, 3 respectively and let us take  $\beta = 0.7$ . For calculating the effect of charges 0, 1 and 2 on charge 3, we need to calculate  $\Phi_{30}$ ,  $\Phi_{31}$  and  $\Phi_{32}$ , which come out: 2.67259, 2.15479 and 1.48268 respectively, in radians. The retarded angle of charge m is 270° – 2 $\Phi_{3m}$  - see Figure (10). So translating into degrees, we get the retarded angles of  $-36.256^{\circ}$ , 23.079° and 100.097° for charges 0, 1 and 2 respectively, which are all smaller than the current angles of those charges.

It is to be mentioned that the solution  $\Phi_{mn}$  of eq. (59) is time independent, meaning that the angle difference between the current position of the charge n and retarded position of the charge m does not depend on time. Because the charges rotate, the only thing which



FIG. 10: Geometrical interpretation for eqs. (57)-(59). The charges rotate on the big circle of radius d. The retarded distance c(t - t') is the big (red) segment on the  $2\Phi_{mn}$  arc, hence equal to  $2d \sin \Phi_{mn}$  according to eq. (57). We also see that the big angle  $2\Phi_{mn}$  (marked in red) equals the sum of the angles  $\omega(t - t')$  and  $2\pi(n - m)/N$  according to eq. (58). Two orthogonal unit vectors (green)  $\hat{\mathbf{r}}(t)$  and  $\hat{\boldsymbol{\varphi}}(t)$  are drawn near charge n, representing the radial and tangential directions of the moving charge.

depends on time are the local unit vectors comoving with the charge  $\hat{\mathbf{r}}(t)$  and  $\hat{\boldsymbol{\varphi}}(t)$  - see Figure (10).

Now the electric field on charge n due to charge m at its retarded position is given by [2, 5]

$$\mathbf{E}_{mn} = \frac{q/N}{4\pi\epsilon_0} \left[ \frac{\widehat{\mathbf{R}}_{mn} - \boldsymbol{\beta}_m}{\gamma^2 R_{mn}^2 (1 - \boldsymbol{\beta}_m \cdot \widehat{\mathbf{R}}_{mn})^3} + \frac{\widehat{\mathbf{R}}_{mn} \times [(\widehat{\mathbf{R}}_{mn} - \boldsymbol{\beta}_m) \times \dot{\boldsymbol{\beta}}_m]}{cR_{mn} (1 - \boldsymbol{\beta} \cdot \widehat{\mathbf{R}}_{mn})^3} \right],$$
(60)

where the 2nd part is the far field which we used in eq. (11) (after replacing **a** by  $\beta c$  and  $1/\sqrt{\epsilon_0\mu_0}$  by c) and the first part is the near field which behaves like  $1/R^2$ .

The quantities appearing in eq. (60) are:  $\boldsymbol{\beta}_m$  is the retarded velocity of charge m (relative to c),  $R_{mn}$  is the distance between the retarded position of charge m and the current position of charge n (and equals to c(t - t') - see Figure (10)), and  $\hat{\mathbf{R}}_{mn}$  is the unit vector pointing from the retarded position of charge m to the current position of charge n.

We need the field at charge n in local components  $\hat{\mathbf{r}}(t)$  and  $\hat{\boldsymbol{\varphi}}(t)$  (see Figure (10)) and we calculate now all the needed quantities to express the electric field  $\mathbf{E}_{mn}$ . So we obtain

$$\widehat{\mathbf{R}}_{mn} - \boldsymbol{\beta}_m = \widehat{\mathbf{r}}(t) [\sin \Phi_{nm} - \beta \sin(2\Phi_{nm})] + \widehat{\boldsymbol{\varphi}}(t) [\cos \Phi_{nm} - \beta \cos(2\Phi_{nm})]$$
(61)

$$\widehat{\mathbf{R}}_{mn} \times \left[ (\widehat{\mathbf{R}}_{mn} - \boldsymbol{\beta}_m) \times \dot{\boldsymbol{\beta}}_m \right] = \omega \beta (\cos \Phi_{nm} - \beta) [\widehat{\mathbf{r}}(\mathbf{t}) \cos \Phi_{nm} - \widehat{\boldsymbol{\varphi}}(t) \sin \Phi_{nm}]$$
(62)

$$\boldsymbol{\beta}_m \cdot \mathbf{R}_{mn} = \beta \cos \Phi_{nm} \tag{63}$$

and

$$R_{mn} = 2d\sin\Phi_{nm} \tag{64}$$

which is easily derived also from Figure (10), because  $R_{mn} = c(t - t')$ , which equals to  $2d \sin \Phi_{nm}$  according to eq. (57). Putting eqs. (61)-(64) in eq. (60) we obtain

$$\mathbf{E}_{mn} = \frac{q/N}{4\pi\epsilon_0} \frac{1}{4d^2 \sin\Phi_{nm}(1-\beta\cos\Phi_{nm})^3} \\ \left[ \frac{\widehat{\mathbf{r}}(t)[\sin\Phi_{nm}-\beta\sin(2\Phi_{nm})] + \widehat{\boldsymbol{\varphi}}(t)[\cos\Phi_{nm}-\beta\cos(2\Phi_{nm})]}{\gamma^2 \sin\Phi_{nm}} + 2\beta^2(\cos\Phi_{nm}-\beta)[\widehat{\mathbf{r}}(\mathbf{t})\cos\Phi_{nm}-\widehat{\boldsymbol{\varphi}}(t)\sin\Phi_{nm}] \right]$$
(65)

With the aid of this field we will calculate the total force acted on a charge by the other charges, but we also need the "self" radiation reaction force. For the most general case, this is given by [2, 5]

$$\mathbf{F}_{\text{self}} = \frac{(q/N)^2}{4\pi\epsilon_0 c^2} \left[ \frac{2}{3} \gamma^2 \ddot{\boldsymbol{\beta}} + 2\gamma^4 (\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}}) \dot{\boldsymbol{\beta}} + \frac{2}{3} \gamma^4 (\boldsymbol{\beta} \cdot \ddot{\boldsymbol{\beta}}) \boldsymbol{\beta} + 2\gamma^6 (\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}})^2 \boldsymbol{\beta} \right].$$
(66)

In our case, of circular motion, the velocity is perpendicular to the acceleration so  $\boldsymbol{\beta} \cdot \dot{\boldsymbol{\beta}} = 0$ , we therefore remain with 2 terms

$$\mathbf{F}_{\text{self}} = \frac{(q/N)^2}{4\pi\epsilon_0 c^2} \left[ \frac{2}{3} \gamma^2 \ddot{\boldsymbol{\beta}} + \frac{2}{3} \gamma^4 (\boldsymbol{\beta} \cdot \ddot{\boldsymbol{\beta}}) \boldsymbol{\beta} \right].$$
(67)

For the circular motion we know that  $\ddot{\boldsymbol{\beta}} = -\omega^2 \boldsymbol{\beta}$ , and using  $\omega = v/d = \beta c/d$ , eq. (67) reduces to

$$\mathbf{F}_{\text{self}} = -\frac{(q/N)^2 \gamma^4}{6\pi\epsilon_0 d^2} \beta^2 \boldsymbol{\beta},\tag{68}$$

showing that the self reaction force is in the direction opposite to the velocity of the charge.

Now we chose n to be the "last" charge, i.e. n = N - 1, hence the restriction in eq. (56) holds, and calculate the total force on it, given by the self force plus the force acted by all other charges (i.e. the Lorentz force):

$$\mathbf{F}_{N-1} = \mathbf{F}_{\text{self}} + (q/N) \sum_{m=0}^{N-2} [\mathbf{E}_{m,N-1} + \mathbf{v}_{N-1}(t) \times \mathbf{B}_{m,N-1}],$$
(69)

where  $\mathbf{v}_{N-1}(t)$  is the velocity of charge N-1 and  $\mathbf{B}_{m,N-1}$  is the retarded magnetic field on charge N-1 due to charge m.

The dumping power on charge N-1 is given by  $\mathbf{v}_{N-1}(t) \cdot \mathbf{F}_{N-1}$ , therefore we do not need the magnetic part, obtaining

$$P_{\text{dump 1 charge}} = \mathbf{v}_{N-1}(t) \cdot \mathbf{F}_{N-1} = \mathbf{v}_{N-1}(t) \cdot \mathbf{F}_{\text{self}} + (q/N) \sum_{m=0}^{N-2} \mathbf{v}_{N-1}(t) \cdot \mathbf{E}_{m,N-1}, \qquad (70)$$

The velocity of the given charge N-1 is in the tangential direction, i.e.  $\mathbf{v}_{N-1}(t) = c\beta \widehat{\boldsymbol{\varphi}}(t)$ , so only the tangential part of eq. (65) affects the power. We obtain

$$P_{\text{dump 1 charge}} = c\beta \left\{ -\frac{(q/N)^2 \gamma^4}{6\pi\epsilon_0 d^2} \beta^3 + \sum_{m=0}^{N-2} \frac{(q/N)^2}{4\pi\epsilon_0} \frac{1}{4d^2 \sin \Phi_{N-1,m} (1-\beta \cos \Phi_{N-1,m})^3} \\ \left[ \frac{\cos \Phi_{N-1,m} - \beta \cos(2\Phi_{N-1,m})}{\gamma^2 \sin \Phi_{N-1,m}} - 2\beta^2 (\cos \Phi_{N-1,m} - \beta) \sin \Phi_{N-1,m} \right] \right\}.$$
(71)

Because the dumping power on one charge does not depend on time and by symmetry is the same for all charges, the total dumping power on the whole system of charges is just N times the above:

$$P_{\text{dump}} = c\beta \left\{ -\frac{q^2 \gamma^4}{6\pi\epsilon_0 N d^2} \beta^3 + \sum_{m=0}^{N-2} \frac{q^2}{4\pi\epsilon_0 N} \frac{1}{4d^2 \sin \Phi_{N-1,m} (1-\beta \cos \Phi_{N-1,m})^3} \left[ \frac{\cos \Phi_{N-1,m} - \beta \cos(2\Phi_{N-1,m})}{\gamma^2 \sin \Phi_{N-1,m}} - 2\beta^2 (\cos \Phi_{N-1,m} - \beta) \sin \Phi_{N-1,m} \right] \right\}.$$
 (72)

The dumping power must be identical with the radiated power, with a minus sign, so to compare them, we may factor out  $-P|_{N=1}$  from eq. (10):

$$P_{\text{dump}} = -P|_{N=1} G_{\text{dump}}(\beta, N), \qquad (73)$$

where

$$G_{\text{dump}}(\beta, N) = \frac{1}{N} - \frac{3}{8N\beta^3 \gamma^4} \sum_{m=0}^{N-2} \frac{1}{\sin \Phi_{N-1,m} (1 - \beta \cos \Phi_{N-1,m})^3} \\ \left[ \frac{\cos \Phi_{N-1,m} - \beta \cos(2\Phi_{N-1,m})}{\gamma^2 \sin \Phi_{N-1,m}} - 2\beta^2 (\cos \Phi_{N-1,m} - \beta) \sin \Phi_{N-1,m} \right].$$
(74)

which clearly shows that for N = 1, the sum is 0, so that  $G_{\text{dump}}(\beta, 1) = 1$  for any  $\beta$ . The numerical calculation of eq. (74) shows identical results with those of G calculated from eq. (47), as shown in Figure 9.

## V. ASYMPTOTIC RESULT FOR MANY CHARGES

To understand how the radiation goes to 0 when the number of charges N goes to infinity, one may try to approximate either G from eq. (47) or  $G_{\text{dump}}$  from eq. (74), for big N. Although  $G_{\text{dump}}$  seems more compact, it is more difficult to handle, and we shall develop Gfor large N.

As mentioned in section 3 (see Figures 3-8), for large N the functions Fc and Fs tend to be harmonic, hence we may approximate them by the first term of their Fourier series.

For a function a(x) with periodicity X, we specify the Fourier coefficients by

$$A_n = \int_0^X \mathbf{a}(x) e^{-i2\pi nx/X} dx \tag{75}$$

and a(x) is represented by its Fourier series

$$a(x) = \frac{1}{X} \sum_{n=-\infty}^{\infty} A_n e^{i2\pi nx/X}$$
(76)

For brevity, to refer to the functions Fc and Fs we call them Fc,s (as in eq. (34)). We know those functions have a periodicity of  $2\pi/N$  in  $\varphi'$  (see eq. (43)), so we define their Fourier coefficients:

$$Ac,s_n = \int_0^{2\pi/N} Fc,s(\varphi')e^{-iNn\varphi'}d\varphi',$$
(77)

and the functions Fc and Fs are expressed as:

$$Fc,s(\varphi') = \frac{1}{2\pi/N} \sum_{n=-\infty}^{\infty} Ac, s_n e^{iNn\varphi'}$$
(78)

Now we calculate the Fourier coefficients  $Ac_{n}$  in eq. (77). The integrand is periodic in  $2\pi/N$ , so increasing the integration interval to  $2\pi$  multiplies the result by N, hence we may express

$$Ac,s_n = \frac{1}{N} \int_0^{2\pi} Fc, s(\varphi') e^{-iNn\varphi'} d\varphi'.$$
(79)

It is to be mentioned that for N = 1, i.e. single charge cyclotron radiation, those Fourier coefficients represent the spectral distribution of the cyclotron radiation, and they compare well with the results in the literature [17–19].

Now by using the definitions of Fc,s (eqs. (20) and (21)) and the property of  $\phi_k$  from eq. (42), we get

Ac,s<sub>n</sub> = 
$$\frac{1}{N} \int_{0}^{2\pi} \sum_{k=0}^{N-1} \text{fc}, s(\phi_0(\varphi' + 2\pi k/N)) e^{-iNn\varphi'} d\varphi'.$$
 (80)

We interchange the sum and the integral and change variable  $\varphi'' = \varphi' + 2\pi k/N$ , obtaining

$$Ac, s_n = \frac{1}{N} \sum_{k=0}^{N-1} e^{-in2\pi k} \int_{2\pi k/N}^{2\pi (1+k/N)} fc, s(\phi_0(\varphi'')) e^{-iNn\varphi''} d\varphi''.$$
(81)

In the above integral, fc,s has a periodicity of  $2\pi$  and  $e^{-iNn\varphi''}$  has a periodicity of  $2\pi/N$ , therefore the integrand is periodic by  $2\pi$ . We may therefore move the integration range to be between 0 and  $2\pi$ , showing that the integral does not depend on k. After renaming  $\varphi''$ to  $\varphi'$  we obtain

$$\operatorname{Ac}_{s_n} = \frac{1}{N} \left( \sum_{k=0}^{N-1} e^{-in2\pi k} \right) \int_0^{2\pi} \operatorname{fc}_{s}(\phi_0(\varphi')) e^{-iNn\varphi'} d\varphi' = \int_0^{2\pi} \operatorname{fc}_{s}(\phi_0(\varphi')) e^{-iNn\varphi'} d\varphi', \quad (82)$$

because  $e^{-in2\pi k} = 1$  for any k. We see that the n Fourier coefficient of Fc,s is actually the Nn Fourier coefficient of fc,s( $\phi_0$ ), which have a  $2\pi$  periodicity. This means that if we represented the fc,s functions by their Fourier components and evaluated Fc,s =  $\sum_{k=0}^{N-1} fc, s(\phi_k)$ , all Fourier components would cancel out except of the nN components, i.e. the 0, N, 2N, etc. The 0 Fourier coefficient is 0, because fc,s have 0 DC level (see eq. (39)), and clearly the second Fourier coefficient of Fc,s, which is the 2N Fourier coefficient of fc,s, is much smaller than the first coefficient for large N, as evident also from Figures (3)-(8).

Therefore, for large N we get the asymptotic Fc,s from its first Fourier coefficient (i.e. coefficients 1 and -1), so that we get from eq. (78):

$$Fc,s(\varphi') \to \frac{N}{2\pi} \left( Ac,s_1 e^{iN1\varphi'} + Ac,s_{-1} e^{iN(-1)\varphi'} \right).$$
(83)

Because Fc,s are real,  $Ac,s_{-1} = Ac,s_1^*$  and we obtain

$$Fc,s(\varphi') \rightarrow \frac{N}{\pi} |Ac,s_1| \cos \left[N\varphi' + \arg(Ac,s_1)\right].$$
 (84)

So we have to calculate the first Fourier coefficients of the Fc and Fs functions,  $Ac_{s_1}$ . From eq. (82) we get

$$Ac,s_1 = \int_0^{2\pi} fc, s(\phi_0(\varphi')) e^{-iN\varphi'} d\varphi', \qquad (85)$$

We change variable to  $\phi_0$  and according to eq. (37) we have  $d\varphi'/d\phi_0 = -1 - p \sin \phi_0$ , so we obtain:

$$Ac, s_1 = \int_{\phi_0(0)}^{\phi_0(0)-2\pi} \frac{hc, s(\phi_0)}{(1+p\sin\phi_0)^3} e^{-iN(-\phi_0+p\cos\phi_0)} d\phi_0(-1-p\,\sin\phi_0), \tag{86}$$

where hc,s is an abbreviation for the functions  $hc(\phi_0)$  and  $hs(\phi_0)$ , defined in eqs. (35) and (36), respectively.

The integrand being periodic on  $2\pi$ , we may shift the limits by any value getting

$$Ac,s_{1} = \int_{0}^{2\pi} \frac{hc,s(\phi_{0})}{(1+p\sin\phi_{0})^{2}} e^{iN\phi_{0}} e^{-iNp\cos\phi_{0}} d\phi_{0}.$$
(87)

This integral is solved by the residue method on the complex plane, in Appendix B, and the results for  $Ac_1$  and  $As_1$  are

$$Ac_{1} = \frac{-2\pi(-i)^{N+1}}{\sqrt{1-p^{2}}} \sum_{n=0}^{\infty} \frac{(-1)^{n} (N/2)^{2n+N+1}}{(n+N+1)!} \sum_{m=0}^{n} \frac{(-1)^{m} (m+1) p^{2(n-m)+N-1}}{(N/2)^{m} (n-m)!} \left[ \left(1+\sqrt{1-p^{2}}\right)^{m+1} - \left(1-\sqrt{1-p^{2}}\right)^{m+1} \right],$$
(88)

and

$$As_{1} = -2\pi (-i)^{N+1} i \sum_{n=0}^{\infty} \frac{(-1)^{n} (N/2)^{2n+N+1}}{(n+N+1)!} \sum_{m=0}^{n} \frac{(-1)^{m} (m+1) p^{2(n-m)+N-1}}{(N/2)^{m} (n-m)!} \left[ \left(1 + \sqrt{1-p^{2}}\right)^{m+1} + \left(1 - \sqrt{1-p^{2}}\right)^{m+1} \right].$$
(89)

We remark that if N is a multiple of 4, the angle of  $Ac_1$  is 90°, and each increment of N

removes 90° from the angle of Ac<sub>1</sub>. Also we see that the angle of As<sub>1</sub> is always bigger by 90° than the angle of Ac<sub>1</sub> and this is visible in Figures (3), (5) and (8). We shall name those angles  $\varphi_c$  and  $\varphi_s$  in the following calculations.

So by setting eqs. (84) in (47), and inverting the integration order between dp and  $d\varphi'$  we obtain:

$$G(\beta, N \to \infty) = \frac{3}{4\pi N \gamma^4 \beta^2} \left[ \int_0^\beta \frac{dp \, p \left(\frac{N}{\pi} |\mathrm{As}_1|\right)^2}{\sqrt{1 - (p/\beta)^2}} \int_0^{2\pi/N} d\varphi' \cos^2(N\varphi' + \varphi_s) + \int_0^\beta dp \, p \left(\frac{N}{\pi} |\mathrm{Ac}_1|\right)^2 \sqrt{1 - (p/\beta)^2} \int_0^{2\pi/N} d\varphi' \cos^2(N\varphi' + \varphi_c) \right].$$

$$(90)$$

The  $d\varphi'$  integrals result in half the integration interval, i.e.  $\pi/N$ , so after simplifying we obtain

$$G(\beta, N \to \infty) = \frac{3}{4\pi N \gamma^4 \beta^2} \frac{N}{\pi} \int_0^\beta dp \, p \left( \frac{|\mathrm{As}_1|^2}{\sqrt{1 - (p/\beta)^2}} + |\mathrm{Ac}_1|^2 \sqrt{1 - (p/\beta)^2} \right). \tag{91}$$

Figure (11) shows the asymptotic results of G for large N according to eqs. (91), (88) and (89), compared with the exact result from eq. (47).

The asymptotic result is much easier calculable than the exact one, and does not require to solve for each step the implicit equation (32), but is still not given by a simple formula.

We will calculate in the next section an asymptotic expression for small  $\beta$ , i.e.  $G(\beta \rightarrow 0, N \rightarrow \infty)$ , and for this case one arrives to a simple formula, as we shall see below.

## VI. ASYMPTOTIC RESULT FOR MANY CHARGES AND LOW VELOCITY

For small  $\beta$ ,  $|p| \ll 1$ , hence  $\sqrt{1-p^2}$  in the denominator of eq. (88) may be set to 1, and  $1 + \sqrt{1-p^2} \approx 2$ , neglecting  $1 - \sqrt{1-p^2}$  in eqs. (88) and (89). So we obtain

$$\operatorname{Ac}_{1}|_{|p|\to 0} = -2\pi(-i)^{N+1} \sum_{n=0}^{\infty} \frac{(-1)^{n} (N/2)^{2n+N+1}}{(n+N+1)!} \sum_{m=0}^{n} \frac{(-1)^{m} (m+1) p^{2(n-m)+N-1}}{(N/2)^{m} (n-m)!} 2^{m+1} \quad (92)$$



FIG. 11: Asymptotic results for G according to eq. (91) versus exact results according to eq. (47) as function of the number of charges N, for different values of  $\beta$ . The asymptotic approximation is accurate even for a few number of charges. As  $\beta$  gets bigger, the difference between asymptotic and exact is more visible for small N.

and

$$As_1|_{|p|\to 0} = iAc_1|_{|p|\to 0}.$$
(93)

Rearranging eq. (92) we obtain

$$\operatorname{Ac}_{1||p|\to 0} = -4\pi (-i)^{N+1} p^{N-1} \sum_{n=0}^{\infty} \frac{(-1)^{n} (N/2)^{2n+N+1} p^{2n}}{(n+N+1)!} \sum_{m=0}^{n} \frac{(m+1)}{(n-m)!} \left(\frac{-4}{Np^{2}}\right)^{m}, \quad (94)$$

and defining  $K \equiv -4/(Np^2)$ , one may express the sum over m in eq. (94) as a derivative with respect to K as follows

$$\partial_K \sum_{m=0}^n \frac{K^{m+1}}{(n-m)!},$$
(95)

and by changing the summation variable m' = n - m, this is written as

$$\partial_K \sum_{m'=n}^0 \frac{K^{n-m'+1}}{m'!} = \partial_K \left[ K^{n+1} \sum_{m'=0}^n \frac{(1/K)^{m'}}{m'!} \right].$$
(96)

We may sum exactly the last sum over m' to obtain

$$\sum_{m'=0}^{n} \frac{(1/K)^{m'}}{m'!} = \exp(1/K) \frac{\Gamma(n+1,1/K)}{n!} \approx \exp(1/K),$$
(97)

where  $\Gamma$  with 2 arguments is the incomplete gamma function. We are interested in small |p|, so for  $|1/K| = Np^2/4 \ll 1$ , we obtain the approximated result given in eq. (97), which means that for small argument, the exponential series in eq. (97) needs very few terms to converge to an exponent. So continuing the calculation started in eq. (96) we obtain

$$\partial_K \left[ K^{n+1} \exp(1/K) \right] = K^n \exp(1/K) [n+1-1/K] \approx (n+1)K^n = (n+1) \left( \frac{-4}{Np^2} \right)^n, \quad (98)$$

where the last approximation used again the fact that  $|1/K| \ll 1$ . Now using the result from eq. (98) in eq. (94), we obtain

$$\operatorname{Ac}_{1}|_{|p|\to 0} = -4\pi (-i)^{N+1} p^{N-1} (N/2)^{N+1} \sum_{n=0}^{\infty} \frac{n+1}{(n+N+1)!} N^{n}.$$
(99)

The above may be summed exactly, obtaining

$$\sum_{n=0}^{\infty} \frac{n+1}{(n+N+1)!} N^n = \frac{N^{N+3}\Gamma(N) - e^N \left(\Gamma(N+2)\Gamma(N+1,N) - N\Gamma(N)\Gamma(N+2,N)\right)}{N^{(2+N)}\Gamma(N)\Gamma(N+2)}.$$
(100)

For large N,  $\Gamma(N+1, N) \approx \frac{1}{2}\Gamma(N+1)$  and  $\Gamma(N+2, N) \approx \frac{1}{2}\Gamma(N+2)$ , hence the expression multiplying the exponent in eq. (100) tents to 0, remaining with

$$\sum_{n=0}^{\infty} \frac{n+1}{(n+N+1)!} N^n \approx \frac{N^{-(2+N)} \left[ N^{N+3} \Gamma(N) \right]}{\Gamma(N) \Gamma(N+2)} = \frac{N}{\Gamma(N+2)} \approx \frac{1}{N!}.$$
 (101)

Putting eq. (101) in (99) we obtain

$$\operatorname{Ac}_{1}|_{|p|\to 0} = -4\pi (-i)^{N+1} p^{N-1} (N/2)^{N+1} \frac{1}{N!} \approx (-i)^{N+1} \sqrt{2\pi N} (e/2)^{N} p^{N-1},$$
(102)

where the last expression has been obtained by using the Stirling approximation, and  $As_1|_{|p|\to 0} = iAc_1|_{|p|\to 0}$  according to eq. (93).

Now we may perform the integral in eq. (91), which becomes for small  $\beta$ 

$$G(\beta \to 0, N \to \infty) = \frac{3 \times 2\pi N (e/2)^{2N}}{4\pi^2 \gamma^4 \beta^2} \int_0^\beta dp \, p^{2N-1} \left( \frac{1}{\sqrt{1 - (p/\beta)^2}} + \sqrt{1 - (p/\beta)^2} \right). \tag{103}$$

Because  $\beta \to 0$ , we may set  $\gamma = 1$ , and we obtain

$$G(\beta \to 0, N \to \infty) = \frac{3N(e/2)^{2N}}{2\pi\beta^2} \frac{\beta^{2N}(N+1)\sqrt{\pi}\,\Gamma(N)}{2\Gamma(N+3/2)}.$$
(104)

For large N,  $\Gamma(N + 3/2)/\Gamma(N) \approx N^{3/2}$ , so we obtain

$$G(\beta \to 0, N \to \infty) = \frac{3N(e/2)^{2N}}{2\pi\beta^2} \frac{\beta^{2N}(N+1)\sqrt{\pi}}{2N^{3/2}} \approx \frac{3(e\beta/2)^{2N}\sqrt{N}}{4\sqrt{\pi}\beta^2}.$$
 (105)

Figure 12 shows the asymptotic results calculated with eq. (105) versus exact results according to eq. (47).

It is to be mentioned that the case of small  $\beta$  and large N analyzed here fits the situations of currents in conducting materials or ion drift currents, mentioned in the introduction. In a conducting loop, the number of charges may be of order of  $10^{23}$ , and  $\beta$  may be of order  $10^{-12}$ , so that  $\beta^{2N}$  results in completely unmeasurable radiated power. In a ion drift device, the number of charges may be of order of  $10^{10}$  and  $\beta$  may be of order  $10^{-6}$ , so that the radiated power is somehow bigger than for the conducting loop, but still unmeasurable.

#### VII. CONCLUSIONS

The purpose of this work was to learn how the radiation from discrete charges vanishes in the continuum steady state limit, i.e. to understand the mechanism by which a system of discrete radiating charges, becomes a non radiative DC current. We used a canonical



FIG. 12: Asymptotic results for G according to eq. (104) versus exact results according to eq. (47) as function of the number of charges N, for different values of  $\beta$ . For  $\beta = 0.05$  the asymptotic result is completely indistinguishable from the exact result, while for  $\beta = 0.1$  and  $\beta = 0.15$  they are almost indistinguishable. For  $\beta > 0.2$  the asymptotic approximation becomes inaccurate.

configuration of charges in uniform circular motion, uniformly spread around a circle.

We found that the log of the power decreases almost linearly with the increase in the number of charges, and arrived to a close form solution to calculate this power if the number of charges is big - see Figure (11).

Specifically, for low velocities, we derived a simple expression for the radiated power. It shows that the radiated power is governed by  $\beta$  at the power of twice the number of charges - see eq. (105) and Figure (12), explaining why the radiation in all the cases considered as DC is unmeasurable.

Although the cyclotron radiation is not the purpose of this work, the formalism used in this work is useful to high energy charged particles applications.

#### Appendix A: Calculation of the average of the functions fc,s

After changing variable  $z = \exp(i\phi_k)$  in eq. (39), one obtains

$$\langle \text{fc,s} \rangle = \frac{1}{2\pi} \frac{2i}{p^2} \oint_C dz \frac{\text{lc,s}(z)}{(z-z_1)^2 (z-z_2)^2} \equiv \frac{1}{2\pi} \frac{2i}{p^2} \oint_C dz \, \text{Lc,s}(z).$$
 (A1)

where C is the counterclockwise unit circle integration contour shown in Figure (A.1) and lc,s are abbreviations for

$$lc(z) \equiv z^2 + 1 \tag{A2}$$

and

$$ls(z) \equiv -i(z^2 - 4bz - 1), \tag{A3}$$

where b is a pure imaginary number defined by

$$b \equiv -0.5ip,\tag{A4}$$

and the 2nd order poles are (expressed in terms of p or b)

$$z_{1,2} = -i\left(1/p \mp \sqrt{(1/p)^2 - 1}\right) = \frac{1}{2}\left(-b^{-1} \pm \sqrt{b^{-2} + 4}\right).$$
 (A5)

where indices 1 and 2 refer to upper and lower signs respectively - see Figure (A.1).

In the last expression of the poles in terms of b, the magnitude under the square root is real and negative, so the square root is understood to be positive pure imaginary.

Also we named the integrand in eq. (A1) Lc,s, defined as

$$Lc,s = \frac{lc,s(z)}{(z - z_1)^2 (z - z_2)^2}$$
(A6)

to ease manipulations. We remark that only  $z_1$  is inside the integration contour (see Figure (A.1)), and its residue is



FIG. A.1: (color online) The complex z plane on which we show: the poles  $z_{1,2}$  in eq (A5), the integration contour C used in eqs. (A1) and (B1), and the integration contour  $C_1$  used in eq. (B6). The poles  $z_{1,2}$  are negative pure imaginary, and  $|z_1| < 1$  and  $|z_2| > 1$ , the integration contour C is on the unit circle and the integration contour  $C_1$  is on a circle of radius smaller than  $|z_1|$ .

$$\operatorname{Res}(L, z1) = \lim_{z \to z_1} \frac{d}{dz} [\operatorname{Lc}, s(z)(z - z_1)^2] = \frac{d}{dz} \left[ \frac{\operatorname{lc}, s(z)}{(z - z_2)^2} \right] \Big|_{z = z_1} = \frac{\operatorname{lc}, s'(z_1)(z_1 - z_2) - 2\operatorname{lc}, s'(z_1)}{(z_1 - z_2)^3}$$
(A7)

Using the relations  $z_1z_2 = -1$  and  $z_1 + z_2 = -2i/p$ , one finds that for both fc and fs cases the nominator of eq. (A7) is 0, and hence

$$\langle fc \rangle = \langle fs \rangle = 0.$$
 (A8)

# Appendix B: Derivation of the first Fourier series coefficient $Ac, s_1$

We change variable  $z = \exp(i\phi_0)$  in eq. (87) to solve this integral on the complex plane, obtaining:

$$Ac, s_1 = \frac{2i}{p^2} \oint_C dz \frac{lc, s(z)}{(z^2 + b^{-1}z - 1)^2} z^N e^{Nb(z+z^{-1})} \equiv \frac{2i}{p^2} \oint_C dz f(z)$$
(B1)

where b is the pure imaginary number defined in eq. (A4), C is the counterclockwise unit circle integration contour (see Figure (A.1)) and the functions lc,s are abbreviations for lc(z)and ls(z) defined in eqs. (A2) and (A3), respectively. To ease on further manipulations we called the integrand f(z).

The integrand has two 2nd order poles,  $z_{1,2}$ , defined in eq. (A5) and only  $z_1$  lies inside the integration contour C - see Figure (A.1). In addition there is an essential singularity at z = 0, because of the  $z^{-1}$  in the exponent.

We first calculate the residue at z = z1. Rewriting f(z)

$$f(z) = \operatorname{Lc}, \operatorname{s}(z) z^{N} e^{Nb(z+z^{-1})}, \qquad (B2)$$

where Lc,s(z) is defined in eq. (A6), we obtain

$$\operatorname{Res}(f, z1) = \lim_{z \to z_1} \frac{d}{dz} [f(z)(z - z_1)^2] = \frac{d}{dz} \left[ \frac{\operatorname{lc,s}(z)}{(z - z_2)^2} z^N e^{Nb(z + z^{-1})} \right] \Big|_{z = z_1}$$
(B3)

which evaluates to

$$\operatorname{Res}(f, z1) = \frac{\operatorname{lc}, s'(z_1)(z_1 - z_2) - 2\operatorname{lc}, s'(z_1)}{(z_1 - z_2)^3} z_1^N e^{Nb(z_1 + z_1^{-1})} + \frac{\operatorname{lc}, s(z_1)}{(z_1 - z_2)^2} \frac{d}{dz} \left[ z^N e^{Nb(z + z^{-1})} \right] \bigg|_{\substack{z=z_1 \\ (B4)}}.$$

We already showed that the first part is 0 - (see eq. (A7)). This is because the integral in eq. (B1) reduces for N = 0 to the integral in eq. (A1) (up to  $2\pi$ ). So we are left with

$$\operatorname{Res}(f, z1) = \frac{\operatorname{lc}, s(z_1)}{(z_1 - z_2)^2} \frac{d}{dz} \left[ z^N e^{Nb(z+z^{-1})} \right] \Big|_{z=z_1} = \frac{\operatorname{lc}, s(z_1)}{(z_1 - z_2)^2} z_1^{N-1} e^{Nb(z_1+z_1^{-1})} N[1+b(z_1-z_1^{-1})].$$
(B5)

By using  $z_1 = -1/z_2$  and  $z_1 + z_2 = -2i/p = -1/b$ , we see that this part is 0 too, hence the contribution of the pole at  $z = z_1$  is 0. We may therefore exclude this pole from the integration contour, and rewrite eq. (B1)

$$Ac,s_{1} = \frac{2i}{p^{2}} \oint_{C_{1}} dz \frac{lc,s(z)}{(z^{2} + b^{-1}z - 1)^{2}} z^{N} e^{Nbz} e^{Nbz^{-1}} = \frac{2i}{p^{2}} \sum_{n=0}^{\infty} \oint_{C_{1}} dz \frac{lc,s(z)}{(z^{2} + b^{-1}z - 1)^{2}} e^{Nbz} \frac{(Nb)^{n}}{n!z^{n-N}}$$
(B6)

where  $C_1$  is the counterclockwise circle of radius smaller than  $|z_1|$ , shown in Figure (A.1). Inside this integration contour we have only the essential singularity at z = 0, and for handling it, we represented the exponent with negative powers of z as a Laurent series. The terms  $n \leq N$  are analytic inside  $C_1$ , hence contribute 0 to the integral, so by changing the summation variable n' = n - (N + 1) and remaining n' to n we obtain

$$Ac,s_{1} = \frac{2i}{p^{2}} \sum_{n=0}^{\infty} \frac{(Nb)^{n+N+1}}{(n+N+1)!} \oint_{C_{1}} dz Lc, s(z) e^{Nbz} \frac{1}{z^{n+1}} \equiv \frac{2i}{p^{2}} \sum_{n=0}^{\infty} \frac{(Nb)^{n+N+1}}{(n+N+1)!} \oint_{C_{1}} dz \ g(z),$$
(B7)

where we used again the definition of Lc,s(z) from eq. (A6), and the integrand has been called g(z), to ease manipulations. We calculate now the residue of g(z)

$$\operatorname{Res}(g,0) = \frac{1}{n!} \frac{d^n}{dz^n} [\operatorname{Lc}, \mathbf{s}(z)e^{Nbz}] \bigg|_{z=0} = \frac{1}{n!} \sum_{m=0}^n \binom{n}{m} \left(\frac{d^m}{dz^m} \operatorname{Lc}, \mathbf{s}(z)\right) \left(\frac{d^{n-m}}{dz^{n-m}} e^{Nbz}\right) \bigg|_{z=0},$$
(B8)

which comes out

$$\operatorname{Res}(g,0) = (Nb)^n \sum_{m=0}^n \frac{(Nb)^{-m}}{(n-m)!m!} \frac{d^m}{dz^m} \operatorname{Lc}_{,\mathrm{s}}(z)|_{z=0},$$
(B9)

We start with Lc. To handle this derivative we express it as

$$\left. \frac{d^m}{dz^m} \mathrm{Lc}(z) \right|_{z=0} = \left. \frac{d^{m+1}}{dz^{m+1}} \frac{z}{1 - b^{-1}z - z^2} \right|_{z=0} \tag{B10}$$

The last rational function is the Fibonacci polynomials generating function, with argument  $b^{-1}$ . Hence the result is (m + 1)! multiplied by the m + 1 Fibonacci polynomial. This may be directly calculated by factorizing the Fibonacci generating function to obtain

$$\frac{d^m}{dz^m} \operatorname{Lc}(z) \bigg|_{z=0} = \frac{1}{\sqrt{b^{-2} + 4}} \frac{d^{m+1}}{dz^{m+1}} \left[ \frac{1}{1 - z_1^{-1}z} - \frac{1}{1 - z_2^{-1}z} \right] \bigg|_{z=0}$$
(B11)

which results in

$$\frac{d^m}{dz^m} \operatorname{Lc}(z) \bigg|_{z=0} = (m+1)! \frac{\left(z_1^{-1}\right)^{m+1} - \left(z_2^{-1}\right)^{m+1}}{\sqrt{b^{-2} + 4}}$$
(B12)

or explicitly

$$\frac{d^m}{dz^m} \operatorname{Lc}(z) \Big|_{z=0} = (m+1)! \frac{\left(b^{-1} + \sqrt{b^{-2} + 4}\right)^{m+1} - \left(b^{-1} - \sqrt{b^{-2} + 4}\right)^{m+1}}{2^{m+1}\sqrt{b^{-2} + 4}}$$
(B13)

The Ls case is handled similarly. We express:

$$\left. \frac{d^m}{dz^m} \mathrm{Ls}(z) \right|_{z=0} = ib \left. \frac{d^{m+1}}{dz^{m+1}} \frac{2 - b^{-1}z}{1 - b^{-1}z - z^2} \right|_{z=0}$$
(B14)

The last rational function is the Lucas polynomials generating function, with argument  $b^{-1}$ . Hence the result is (m + 1)! multiplied by the m + 1 Lucas polynomial. This may be directly calculated by factorizing the Lucas generating function to obtain

$$\left. \frac{d^m}{dz^m} \mathrm{Ls}(z) \right|_{z=0} = ib \left. \frac{d^{m+1}}{dz^{m+1}} \left[ \frac{1}{1 - z_1^{-1}z} + \frac{1}{1 - z_2^{-1}z} \right] \right|_{z=0}$$
(B15)

which results in

$$\left. \frac{d^m}{dz^m} \mathrm{Ls}(z) \right|_{z=0} = ib(m+1)! \left[ \left( z_1^{-1} \right)^{m+1} + \left( z_2^{-1} \right)^{m+1} \right]$$
(B16)

or explicitly

$$\left. \frac{d^m}{dz^m} \mathrm{Ls}(z) \right|_{z=0} = ib(m+1)! \frac{\left(b^{-1} + \sqrt{b^{-2} + 4}\right)^{m+1} + \left(b^{-1} - \sqrt{b^{-2} + 4}\right)^{m+1}}{2^{m+1}} \tag{B17}$$

By using eqs. (B13), (B17), (B9) and (B7) and replacing b = -0.5ip we obtain a closed form expression for Ac<sub>1</sub> and Ac<sub>2</sub>, given in eqs. (88) and (89) respectively.

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