Coherent emission from a bunched electron beam: superradiance and stimulated-superradiance in a uniform and tapered wiggler FEL

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We outline fundamental coherent radiation processes from a charge particles beam: Spontaneous Superradiance (SR), Stimulated Superradiance (ST-SR), and in the context of undulator radiation: Tapering-Enhanced Superradiance (TES) and Tapering-Enhanced Stimulated Superradiance Amplification (TESSA). Both single bunch and periodic bunching (in phasor and spectral Fourier frequency formulations) are considered in a model of radiation mode expansion.

I. INTRODUCTION

In the context of radiation emission from an electron beam, Dicke's superradiance (SR) [1] is the enhanced radiation emission from a pre-bunched beam. Stimulated Superradiance (ST-SR) is the further enhanced emission of the bunched beam in the presence of a phase-matched radiation wave. These coherent spontaneous emission processes of an electron bunch were analyzed for synchrotron and undulator radiation in the framework of radiation field mode-excitation theory [2]. In the nonlinear saturation regime of a radiating bunched beam the synchronism of the slowing down bunched beam with an injected co-propagating radiation wave may be sustained by wiggler tapering in a process of Taper-Enhanced Superradiance (TES) and Taper-Enhanced Stimulated Superradiance Amplification (TESSA) [3]. Same processes are instrumental also in enhancing the radiative emission in the tapered wiggler section of seeded FEL [4, 5]. Considering these radiation emission concepts provide guidelines for better design of high power FELs and improved tapering strategy for enhancing the power of seeded short wavelength FELs.

In Sect. II we show the general expressions for random spontaneous emission, super-radiant emission and stimulated-superradiant emission in a general spectral (Fourier transform) presentation of Maxwell equations. We employ the formulation specifically to the case of undulator radiation (UR) emission by a single bunch or a finite duration pulse of periodic bunches ("bunch train"). In Sect. III the analysis of spontaneous super-radiance (SR) and "zero-order" stimulated super-radiance (ST-SR) is carried out in phasor formulation for the steady state case of a periodically bunched electron beam (namely, an infinite train of bunches), within the approximation of negligible energy loss of the radiating e-beam. In Sect. IV we extend the phasor analysis of superradiance and zero order stimulated super-radiance to the case of a tapered wiggler (TES and TESSA). The dynamics of electron interaction and energy loss due to interaction with the radiation is neglected, assuming that ideally the wiggler tapering rate exactly matches the energy loss of the perfectly bunched mono-energetic beam. We compare for this case the relative intensity of the zero order TESSA and the TES powers and its scaling with interaction length. This comparison is of interest in connection to the tapered wiggler section of a seed injected FEL. In Sect. V the dynamics of the interaction of the electron beam bunch with the radiation is taken into account by inclusion of the electron force equations.

II. SUPERRADIANCE AND STIMULATED SUPERRADIANCE OF SPONTANEOUS EMISSION

As a starting point we review the theory of superradiant (SR) and stimulated superradiant (ST-SR) emission from free electrons in a general radiative emission process. In this section we use a spectral formulation, namely, all fields are given in the frequency domain as Fourier transforms of the real time-dependent fields:

$$\breve{A}(r,\omega) = \int_{-\infty}^{\infty} A(r,t)e^{i\omega t}dt \tag{1}$$

We use the radiation modes expansion formulation of [2], where the radiation field is expanded in terms of an orthogonal set of eigenmodes in a waveguide structure or in free space (eg. Hermite-Gaussian modes):

$$\{\tilde{\boldsymbol{E}}_q(\mathbf{r}), \tilde{\boldsymbol{H}}_q(\mathbf{r})\} = \{\tilde{\boldsymbol{\mathcal{E}}}_q(\mathbf{r}_\perp), \tilde{\boldsymbol{\mathcal{H}}}_q(\mathbf{r}_\perp)\}e^{ik_{qz}z} \tag{2}$$

$$\breve{\mathbf{E}}(\mathbf{r},\omega) = \sum_{\pm q} \breve{C}_q(z,\omega)\tilde{\boldsymbol{\mathcal{E}}}_q(\mathbf{r}) \tag{3}$$

$$\breve{\mathbf{H}}(\mathbf{r},\omega) = \sum_{\pm q} \breve{C}_q(z,\omega)\tilde{\boldsymbol{\mathcal{H}}}_q(\mathbf{r}) \tag{4}$$

The electric/magnetic fields representing the transverse profile of the mode are named $\tilde{\boldsymbol{\mathcal{E}}}$ and $\tilde{\boldsymbol{\mathcal{H}}}$ and are supposed to be frequency independent. The excitation equations of the mode amplitudes is:

$$\frac{d\breve{C}_q(z,\omega)}{dz} = \frac{-1}{4\mathcal{P}_q}\int \breve{\mathbf{J}}(\mathbf{r},\omega)\cdot\tilde{\boldsymbol{E}}_q^*(\mathbf{r})d^2\mathbf{r}_\perp. \tag{5}$$

where the current density $\breve{\mathbf{J}}(\mathbf{r},\omega)$ is the Fourier transform of $\mathbf{J}(\mathbf{r},t)$.

The above is formally integrated and given in terms of the initial mode excitation amplitude and the currents

$$\breve{C}_q(z,\omega) - \breve{C}_q(0,\omega) = -\frac{1}{4\mathcal{P}_q}\int \breve{\mathbf{J}}(\mathbf{r},\omega)\cdot\tilde{\boldsymbol{E}}_q^*(\mathbf{r})dV, \tag{6}$$

where

$$\mathcal{P}_q = \frac{1}{2}Re\iint(\tilde{\boldsymbol{\mathcal{E}}}_q\times\tilde{\boldsymbol{\mathcal{H}}}_q)\cdot\hat{e}_z d^2\mathbf{r}_\perp = \frac{|\tilde{\boldsymbol{\mathcal{E}}}_q(\mathbf{r}_\perp=0)|^2}{2Z_q}A_{em\,q}, \tag{7}$$

where Z_q is the mode impedance (in free space $Z_q = \sqrt{\mu_0/\epsilon_0}$), and for a narrow beam, passing on axis near $\mathbf{r}_\perp = 0$, Eq. (7) defines the mode effective area $A_{em\,q}$ in terms of the field of the mode on axis $\tilde{\boldsymbol{\mathcal{E}}}_q(\mathbf{r}_\perp = 0)$. For Fourier transformed fields we define the total spectral energy (per unit of angular frequency) based on Parseval theorem as

$$\frac{dW}{d\omega} = \frac{2}{\pi}\sum_q \mathcal{P}_q|\breve{C}_q(\omega)|^2, \tag{8}$$

This is the definition when using only positive frequencies ($0 < \omega < \infty$). Considering now one single mode q,

$$\frac{dW}{d\omega} = \frac{2}{\pi}\mathcal{P}_q|\breve{C}_q(\omega)|^2, \tag{9}$$

For a particulate current (an electron beam):

$$J(\mathbf{r},t) = \sum_{j=1}^{N} -e\mathbf{v}_j(t)\delta(\mathbf{r}-\mathbf{r}_j(t)) \tag{10}$$

the field amplitude increment appears as a coherent sum of contributions (energy wavepackets) from all the electrons in the beam:

$$\breve{C}_q^{out}(\omega) - \breve{C}_q^{in}(\omega) \equiv \sum_{j=1}^{N} \Delta\breve{C}_{qj}(\omega) = -\frac{1}{4\mathcal{P}_q}\sum_{j=1}^{N}\Delta\breve{\mathcal{W}}_{qj} \tag{11}$$

$$\Delta\breve{\mathcal{W}}_{qj} = -e\int_{-\infty}^{\infty} \mathbf{v}_j(t)\cdot\tilde{\boldsymbol{E}}_q^*(\mathbf{r}_j(t))e^{i\omega t}dt \tag{12}$$

The contributions can be split into a spontaneous part (independent of the presence of radiation field) and stimulated (field dependent) part:

$$\Delta\breve{\mathcal{W}}_{qj} = \Delta\breve{\mathcal{W}}_{qj}^0 + \Delta\breve{\mathcal{W}}_{qj}^{st}. \tag{13}$$

We do not deal in this section with stimulated emission, but indicate that in general the second term $\Delta\breve{\mathcal{W}}_{qj}^{st}$ is a function of $\breve{C}_q(z)$ through $\mathbf{r}_j(t)$ and therefore $\mathbf{v}_j(t)$ cannot be calculated explicitly from the integral (12). Its calculation requires solving the electron force equations and the differential equation (5). In the context of conventional FEL in the linear regime, $\Delta\breve{C}_{qj}^{st}$ is proportional to the input field, i.e. proportional to $\breve{C}_q^{in}$, and in this case the solution of (5) results in the exponential gain expression of conventional FEL.

Assuming a narrow cold beam where all particles follow the same trajectories, we may write $\mathbf{r}_j(t) = \mathbf{r}_j^0(t-t_{0j})$ and $\mathbf{v}_j(t) = \mathbf{v}_j^0(t-t_{0j})$, change variable $t' = t - t_{0j}$ in Eq. (12) (see MOP078), so that the spontaneous emission wavepacket contributions are identical, except for a phase factor corresponding to their injection time t_{0j}:

$$\Delta\breve{\mathcal{W}}_{qj}^0 = \Delta\breve{\mathcal{W}}_{qe}^0 e^{i\omega t_{0j}} \tag{14}$$

where

$$\Delta\breve{\mathcal{W}}_{qe}^0 = -e\int_{-\infty}^{\infty} v_e^0(t)\cdot\tilde{\boldsymbol{E}}_q^*(r_e^0(t))e^{i\omega t}dt. \tag{15}$$

The radiation mode amplitude at the output is composed of a sum of wavepacket contributions including the input field contribution (if any):

$$\breve{C}_q^{out}(\omega) = \breve{C}_q^{in}(\omega) + \Delta\breve{C}_{qe}^0(\omega)\sum_{j=1}^{N} e^{i\omega t_{0j}} + \sum_{j=1}^{N}\Delta\breve{C}_{qj}^{st} =$$
$$\breve{C}_q^{in}(\omega) - \frac{1}{4\mathcal{P}_q}\Delta\breve{\mathcal{W}}_{qe}^0\sum_{j=1}^{N} e^{i\omega t_{0j}} - \frac{1}{4\mathcal{P}_q}\sum_{j=1}^{N}\Delta\breve{\mathcal{W}}_{qj}^{st} \tag{16}$$

so that the total spectral radiative energy from the electron pulse is

$$\frac{dW_q}{d\omega} = \frac{2}{\pi}\mathcal{P}_q\left|\breve{C}_q^{out}(\omega)\right|^2 =$$
$$\frac{2}{\pi}\mathcal{P}_q\left\{\left|\breve{C}_q^{in}(\omega)\right|^2 + \left|\Delta C_{qe}^{(0)}(\omega)\right|^2\left|\sum_{j=1}^{N} e^{i\omega t_{oj}}\right|^2 + \right.$$
$$\left[\breve{C}_q^{in\,*}(\omega)\Delta C_{qe}^{(0)}(\omega)\sum_{j=1}^{N} e^{i\omega t_{oj}} + c.c.\right] +$$
$$\left.\left[\breve{C}_q^{in\,*}(\omega)\sum_{j=1}^{N}\Delta C_{qj}^{st}(\omega) + c.c.\right] + \left|\sum_{j=1}^{N}\Delta C_{qj}^{st}(\omega)\right|^2\right\} \equiv$$
$$\left(\frac{dW_q}{d\omega}\right)_{in} + \left(\frac{dW_q}{d\omega}\right)_{sp/SR} + \left(\frac{dW_q}{d\omega}\right)_{ST-SR} + \left(\frac{dW_q}{d\omega}\right)_{st}. \tag{17}$$

The first term in the {} parentheses represents the input field, given the subscript "in". The second term is the spontaneous emission, which may be random beam spontaneous emission or superradiant in case that all contributions add in phase, hence given the subscript "sp/SR". The third term has a very small value (averages to 0) if the contributions add randomly, so it is relevant only if the electrons of the beam enter in phase with the injected radiation wave. It is thus dependent on the mode complex amplitude $\breve{C}_q^{in}$, and therefore it marked by the subscript "ST-SR", i.e. "zero-order" stimulated superradiance. The last 2 terms in the {} parentheses represent stimulated emission and we will not consider them further in this work. Fig. 1(a) and (b) represent

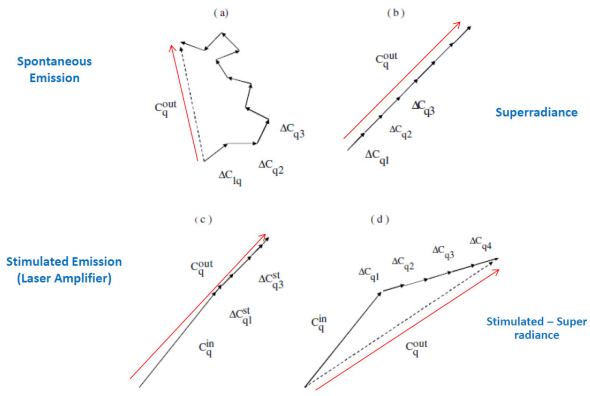

FIG. 1: Different cases of radiation: (a) spontaneous emission, (b) superradiance, (c) stimulated emission and (d) stimulated superradiance.

in $\breve{C}_q$ complex plane the random and constructive contributions of the radiation wavepackets to conventional spontaneous emission and superradiance emission respectively. These corresponds to the second term in Eq. (17) where in 1(a) the wavepackets interfere randomly and in 1(b), constructively in phase. Fig. 1(d) represents the third term in Eq. (17) where the coherent constructive interference of a prebunched beam interferes with the input field with some phase offset. Fig. 1(c) represents regular stimulated emission from a randomly injected electron beam (regular FEL).

When the electrons in the beam are injected at random in a long pulse, and averaging the second term in Eq. (17), the uncorrelated mixed terms cancel out and one obtains the conventional shot-noise driven spontaneous emission [2, 9].

$$\left(\frac{dW_q}{d\omega}\right)_{sp} = \frac{1}{8\pi\mathcal{P}_q}\left|\Delta\breve{\mathcal{W}}_{qe}^{(0)}\right|^2 N \tag{18}$$

Only when the electrons are bunched into a pulse shorter than an optical period $\omega(\Delta t_{0i} - t_0) \ll \pi$ one gets enhanced superradiant spontaneous emission, in which case all the terms in the bracket of the third term of Eq. (17) add up constructively in phase $\sum_{j=1}^{N} e^{i\omega t_{oj}} = Ne^{i\omega t_o}$ resulting

$$\left(\frac{dW_q}{d\omega}\right)_{sp} = \frac{1}{8\pi\mathcal{P}_q}\left|\Delta\breve{\mathcal{W}}_{qe}^{(0)}\right|^2 N^2 = \left\langle\left(\frac{dW_q}{d\omega}\right)\right\rangle_{sp} N \tag{19}$$

Figure 1(d) displays a process of of stimulated superradiance: all electrons oscillate in phase, but because a radiation mode of distinct phase is injected in, the third term in Eq. (17) will contribute positive or negative energy, depending whether the electron bunch oscillates in phase or out of phase with the input radiation field. If the phase of the electron bunch relative to the wave is φ, then the third term in Eq. (17) represents stimulated superradiance spectral energy:

$$\left(\frac{dW_q}{d\omega}\right)_{ST-SR} = -\frac{1}{2\pi}\left|\breve{C}_q^{in}\right|\left|\Delta\breve{\mathcal{W}}_{qe}^{(0)}\right| N\cos\varphi. \tag{20}$$

At this point we extend the analysis to include partial bunching, namely electron beam bunches of finite duration and arbitrary bunch-shape function. One can characterize the distribution

of electron entrance times toj of the electron bunch by means of a normalized bunch-shape function $f(t_0' - t_0) = i(t_0' - t_0)/(eN)$, where $i(t)$ is the e-beam bunch current, and t_0 is the bunch center entrance time:

$$\int_{-\infty}^{\infty} f(t_0' - t_0)dt_0' = 1. \tag{21}$$

Then the summation over t_{0j} may be substituted by integration over entrance times t_0':

$$\sum_{j=1}^{N} e^{i\omega t_{oj}} = N \int f(t_0' - t_0)e^{i\omega t_0'}dt_0' = Ne^{i\omega t_0} M_b(\omega), \tag{22}$$

where

$$M_b(\omega) = \frac{1}{N}\left\langle \sum_{j=1}^{N} e^{i\omega t_{0j}} \right\rangle = \int f(t)e^{i\omega t}dt, \tag{23}$$

is the Fourier transform of the bunch-shape function, i.e. the bunching amplitude at frequency ω. It modifies Eqs. (19) and (20) to

$$\left(\frac{dW_q}{d\omega}\right)_{sp} = \frac{1}{8\pi\mathcal{P}_q}\left|\Delta\breve{\mathcal{W}}_{qe}^{(0)}\right|^2 |M_b|^2 N^2 \tag{24}$$

and

$$\left(\frac{dW_q}{d\omega}\right)_{ST-SR} = -\frac{1}{2\pi}\left|\breve{C}_q^{in}\right|\left|\Delta\breve{\mathcal{W}}_{qe}^{(0)}\right| |M_b| N\cos\varphi. \tag{25}$$

In conditions of perfect bunching $f(t) = \delta(t)$, and consequently $M_b = 1$, Eqs. (19) and (20) are restored.

For the case of undulator radiation we specify for each electron:

$$\mathbf{v}_j(t) = Re\left[\tilde{\mathbf{v}}_{\perp j}e^{-ik_w z_j(t)}\right] \tag{26}$$

where

$$\tilde{\mathbf{v}}_{\perp j} = \frac{c\tilde{\mathbf{a}}_w}{\gamma_j} = \frac{ce\hat{z}\times\tilde{\mathbf{B}}_w}{\gamma_j m k_w} \tag{27}$$

where $\tilde{B}_w$ is the complex amplitude of the undulator periodic magnetic field. Assume that the electron beam is narrow enough so that all electrons experience the same field when interacting with the mode

$$\tilde{\boldsymbol{\mathcal{E}}}_q(\mathbf{r}_j^0(t)) = \tilde{\boldsymbol{\mathcal{E}}}_q(\mathbf{r}_\perp = 0)e^{ik_{qz}z_j^0(t)} \tag{28}$$

where $z_j^0(t) = v_z(t - t_{0j})$, and $\mathbf{r}_\perp$ is the transverse coordinates vector of the electron beam. Substituting this and Eq (26) in (14) one obtains

$$\Delta\breve{\mathcal{W}}_{qj}^0 = -e\frac{\tilde{\mathbf{v}}_{\perp 0}\cdot\tilde{\boldsymbol{\mathcal{E}}}_q^*}{2v_z}L\,\mathrm{sinc}(\theta L/2)e^{i\theta L/2}e^{i\omega t_{0j}}, \tag{29}$$

where $L = N_w\lambda_w$ is the interaction length ($\lambda_w = 2\pi/k_w$), $\mathrm{sinc}(x) = \sin x/x$, and $\theta(\omega)$, the detuning parameter, is defined by

$$\theta(\omega) = \frac{\omega}{v_z} - k_{zq}(\omega) - k_w. \tag{30}$$

The detuning function $\mathrm{sinc}(\theta L/2)$ attains its maximum value at the synchronism frequency ω_0 defined by

$$\theta(\omega_0) = \frac{\omega_0}{v_z} - k_{zq}(\omega_0) - k_w = 0. \tag{31}$$

Near synchronism

$$\theta(\omega)L \simeq (\omega - \omega_0)t_s = 2\pi\frac{\omega - \omega_0}{\Delta\omega}. \tag{32}$$

where

$$t_s = \frac{2\pi}{\Delta\omega} = \frac{L}{v_z} - \frac{L}{v_{gq}} \tag{33}$$

is the wave packet slippage time and $v_{gq} = d\omega/dk_{zq}$ at ω_0 is the group velocity of the mode. In free space $k_{zq} = \omega/c$, $v_{gq} = c$, and the solution of (31) is the well known FEL frequency:

$$\omega_0 = \frac{ck_w}{1/\beta_z - 1} \simeq 2\gamma_z^2 ck_w \tag{34}$$

where

$$\gamma_z^2 = \frac{\gamma^2}{1 + \overline{a}_w^2} \tag{35}$$

And $\overline{a}_w$ is the one period r.m.s. average of $a_w(z)$. It is equal to the amplitude a_w in the case of a helical wiggler and to $a_w/2$ in a linear wiggler. The second part of Eq. (34) applies for an ultra-relativistic beam ($\beta \simeq 1$). In this limit

$$\Delta\omega = \frac{\omega_0}{N_w} \tag{36}$$

When substituting (29) into (19) one obtains the expression of UR superradiance from a tight single bunch into a single mode q:

$$\left(\frac{dW_q}{d\omega}\right)_{SR} = \frac{N^2e^2Z_q}{16\pi}\left(\frac{\overline{a}_w}{\beta_z\gamma}\right)^2\frac{L^2}{A_m}\mathrm{sinc}^2(\theta L/2) \tag{37}$$

We now extend the analysis to the case of spontaneous emission from a finite train of bunches. Following the formulation of [2], we consider a train of N_M identical bunches (neglecting shot noise) separated in time $T_b \equiv 2\pi/\omega_b$ apart. The arrival times of bunch k is

$$t_{0k} = [k - (N_M/2)]2\pi/\omega_b \tag{38}$$

In each bunch there are N_b electrons, the electron j being at time interval $\Delta t_j \ll T_b$ relative to the center of the bunch. For the case of interest of UR we obtain the general expression for spontaneous SR and ST-SR spectral energy of a finite train of periodic bunches:

$$\left(\frac{dW_q}{d\omega}\right)_{SR} = \frac{N^2e^2Z_q}{16\pi}\left(\frac{\overline{a}_w}{\beta_z\gamma}\right)^2\frac{L^2}{A_m}|M_b(\omega)|^2 |M_M(\omega)|^2\,\mathrm{sinc}^2(\theta L/2) \tag{39}$$

and the stimulated superradiant term is

$$\left(\frac{dW_q}{d\omega}\right)_{ST-SR} = |\breve{C}_q^{in}(\omega)|\frac{Ne}{2\pi}\left(\frac{\overline{a}_w}{\beta_z\gamma}\right)\sqrt{\frac{2Z_q\mathcal{P}_q}{A_{em\,q}}}L\,|M_b(\omega)|\,|M_M(\omega)|\,\mathrm{sinc}(\theta L/2)\cos(\varphi - \theta L/2) \tag{40}$$

where

$$M_M(\omega) = \frac{\sin(N_M\pi\omega/\omega_b)}{N_M\sin(\pi\omega/\omega_b)} \tag{41}$$

and $N = N_M N_b$.

III. SINGLE FREQUENCY (PHASOR) FORMULATION

In the limit of a continuous train of microbunches or a long macropulse $N_M \gg 1$, the grid function $M_M(\omega)$ behaves like a comb of delta functions and narrows the spectrum of the prebunched beam SR and ST-SR Undulator Radiation to harmonics of the bunching frequencies $\omega = n\omega_b$. Instead of spectral energy, one can then evaluate the average radiation power output by integrating the spectral energy expressions over frequency and dividing by the pulse duration: $T_M = N_M 2\pi/\omega_b$. Alternatively, for an infinite periodic train of identical bunches one may use directly a steady-state single frequency (phasor) formulation. It is to be mentioned that in this case the radiation frequency ω must be equal to the bunch frequency $\omega = \omega_0 = \omega_b$, otherwise there will not be any steady-state interaction between them. The radiation mode excitation equations in the phasor formulation of the radiation fields $\{\tilde{\boldsymbol{E}}_q(\mathbf{r}), \tilde{\boldsymbol{H}}_q(\mathbf{r})\}$ is the same as Eqs. (2)-(5) with $\tilde{C}_q(\omega_0) \equiv \tilde{C}_q$ replacing $\breve{C}_q(\omega)$, and the spectral energy expression (8) replaced by the total steady state radiation power. As in [6–8], we take a model of a periodically modulated e-beam current of a single frequency ω_0:

$$I(z,t) = I_0\{1 + Re[\tilde{M}_b e^{-i\omega_0(t - z/v_z)}]\} \tag{42}$$

Assuming the beam has a normalized transverse profile distribution $f(\mathbf{r}_\perp)$. The transverse current density in the wiggler is:

$$\mathbf{J}_\perp(\mathbf{r},\omega_0) = \frac{\tilde{I}_{m\perp}\hat{\mathbf{e}}_\perp}{2} f(\mathbf{r}_\perp) e^{i(\omega_0/v_z - k_w)z} \tag{43}$$

Where:

$$\tilde{I}_{m\perp}\hat{\mathbf{e}}_\perp = I_0 \tilde{M}_b \frac{\tilde{\boldsymbol{\beta}}_w}{\beta_z} \tag{44}$$

Writing now the excitation equation in phasor formulation we obtain the superradiant and stimulated superradiant powers:

$$P_{SR}(z) = \frac{1}{32} Z_q |\tilde{I}_{m\perp}|^2 F^2 \frac{z^2}{A_{em\,q}} \operatorname{sinc}^2(\theta z/2) \tag{45}$$

and

$$P_{ST-SR}(z) = \frac{1}{4} |\tilde{C}_q(0)||\tilde{I}_{m\perp}||E_\perp(0)| F z \cos(\varphi_{rb0} - \theta z/2) \operatorname{sinc}(\theta z/2) \tag{46}$$

where

$$\varphi_{rb0} = \varphi_q(0) - \varphi_{b0} \tag{47}$$

is the phase difference between the radiation field phase $\varphi_q(0)$ and the bunching current phase φ_{b0} at the entrance to the wiggler.

IV. TAPER ENHANCED SUPERRADIANCE (TES) AND TAPER ENHANCED STIMULATED SUPERRADIANCE AMPLIFICATION (TESSA)

We now extend our model to the case of a continuously bunched electron beam interacting with a strong radiation field in an undulator, so that the electron beam loses an appreciable portion of its energy in favor of the radiation field. This case is particularly

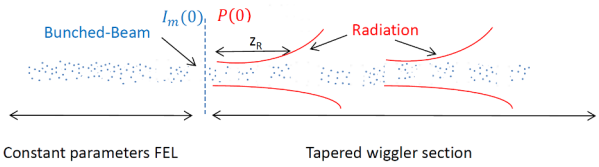

FIG. 2: Schematics of seed-injected FEL followed by a tapered wiggler: at saturation point, at the end of the constant parameters wiggler, the partially bunched e-beam and the amplified radiation wave are injected into a tapered wiggler section, where further radiation energy is extracted out of the bunched beam.

relevant to the case of seed injected tapered wiggler FEL. In this case (see Figure 2) the tapered wiggler section would emit both TES and TESSA radiation. The input field amplitude and phase relative to the beam bunching that enter into the TESSA process are determined in this case by the gain and saturation processes in the constant wiggler parameters FEL section preceding the tapered wiggler section.

For a tapered wigler $k_w = k_w(z)$, $\overline{a}_w = \overline{a}_w(z)$ we extend the definition of the detuning parameter (30):

$$\theta(z) = k_0 \left[\beta_z^{-1} - 1\right] - k_w(z). \tag{48}$$

where $k_0 = \omega_0/c$. The synchronism condition $\theta(z) = 0$ defines the z-dependent synchronism energy of the electron beam:

$$\gamma_r(z) = \sqrt{\frac{1+\overline{a}_w^2(z)}{1-(1+k_w(z)/k_0)^{-2}}} \simeq \sqrt{\frac{1+\overline{a}_w^2(z)}{2}\frac{k_0}{k_w(z)}}, \tag{49}$$

where the second part of the equation corresponds to the ultra-relativistic beam limit.

Assuming that the bunched electrons get trapped, so that in the presence of fields they stay with energy close to the synchronism energy $\gamma_r(z)$ we write

$$\gamma = \gamma_r + \delta\gamma \tag{50}$$

and therefore

$$\theta = \left.\frac{d\theta}{d\gamma}\right|_{\gamma_r} \delta\gamma = -\frac{k_0}{\beta_{zr}^3 \gamma_{zr}^2 \gamma_r}\delta\gamma, \tag{51}$$

The phase of the bunched beam relative to the ponderomotive wave is then:

$$\varphi(z) = \int_0^z \left[-\frac{k_0}{\beta_{zr}^3\gamma_{zr}^2\gamma_r}\delta\gamma\right] dz' + \varphi_{b0} = \int_0^z \theta^E(z')dz' + \varphi_{b0}, \tag{52}$$

Consequently,

$$\tilde{C}_q(z) = \tilde{C}_q(0) - \frac{1}{8} I_0 F \frac{|\breve{\boldsymbol{\mathcal{E}}}_q(0)|}{\mathcal{P}_q} e^{i\varphi_{b0}} \int_0^z \frac{\overline{a}_w(z')}{\gamma(z')\beta_z(z')} |M_b(z')| e^{i\int_0^{z'}\theta^E(z'')dz''} dz' \tag{53}$$

Of course, in order to know $\delta\gamma(z, E(z))$ and consequently $\theta^E(z)$ one must solve the force equation for the bunched electron beam dynamics in the buckets of the slowing down ponderomotive potential, as in the next section. Here we consider only the optimal tapering strategy, such that the beam energy loss rate matches the wiggler tapering, so that the detuning parameter stays constant: $\theta^E(z) = \theta^E(0)$. If we also assume as in [6] that the bunching amplitude and amplitude coefficient in the integrand are approximately constant, then the equation is integrable, resulting in:

$$\tilde{C}_q(z) = \tilde{C}_q(0) - \frac{\tilde{I}_{m\perp}}{8\mathcal{P}_q} |\tilde{\boldsymbol{\mathcal{E}}}_q(0)| F z e^{i\theta_0^E z/2} \operatorname{sinc}\left(\theta_0^E z/2\right) \tag{54}$$

Similarly to Eqs. (45) and (46) for SR/ST-SR we get then for a tapered wiggler with tapering matched:

$$P(z) = P(0) + P_{TES}(z) + P_{TESSA}(z) \tag{55}$$

where for $\theta_0^E = 0$ and phase matched bunched current and radiation field $\varphi_{rb0} = 0$ [6]

$$P_{TES}(z) = \frac{1}{32} Z_q |\tilde{I}_{m\perp}|^2 F^2 \frac{z^2}{A_{em\,q}} \tag{56}$$

and

$$P_{TESSA}(z) = \frac{1}{4} |\tilde{I}_{m\perp}| \sqrt{\frac{2Z_q}{A_{em\,q}}} \sqrt{P_{in}} F z \tag{57}$$

The ratio between the two contributions to the radiation power is proportional to $E_{in}(0)$, and inverse proportional to the distance z. It is shown in Figure 3 for different initial power at $z = z_0$. Initially the TESSA power dominates the TES power, but evidently, for long interaction length the TES power that grows like z^2 exceeds the TESSA power that grows like z. At the beginning stages of interaction in the tapered wiggler the TESSA power may be significantly higher than the TES power if the initial radiation power P_{in} injected into the tapered section is large enough. This balance is demonstrated in Figure 3 for the parameters of LCLS [10].

V. DYNAMICS OF A PERIODICALLY BUNCHED ELECTRON BEAM INTERACTING WITH RADIATION FIELD IN A GENERAL WIGGLER

In this section we extend the analysis of SR and ST-SR in undulator radiation of a periodically bunched beam, that was presented

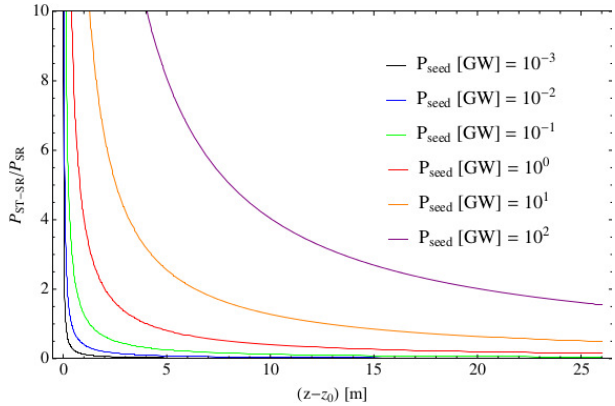

FIG. 3: Ratio of 0-order TESSA to TES for different initial power at $z = z_0$.

in sections III, IV based on radiation mode excitation and phasor formulations, and we add the dynamics of the electrons under interaction with the radiation wave. Solving now a steady-state problem, we assume that the periodically bunched beam is composed of all identical bunches (namely, shot-noise and finite pulse effects are neglected). The bunches are tightly bunched, so they can be modeled as Dirac delta functions. They all experience the same force equation and have the same trajectories as macro-particles of charge $Q_b = -eN_b$ and the time interval between two consecutive injected bunches is $T_0 \equiv 2\pi/\omega_0$, therefore

$$\mathbf{J}(\mathbf{r},t) = Q_b \mathbf{v}_e(t)\delta(\mathbf{r}_\perp) \sum_{n=-\infty}^{\infty} \delta[z - z_e(t - nT_0 - t_0)] \tag{58}$$

The phasor mode excitation equation (5) of any harmonic of the radiation emitted by the current (58) is applied with these simplifying assumptions for calculating the radiation power. For the tight bunch train model of Eq (58) the density of electrons per unit volume is defined

$$n(\mathbf{r},t) = N_b \delta(\mathbf{r}_\perp) \sum_j \delta\left[z - \int_{t_{oj}}^{t} v_z(t')dt'\right], \tag{59}$$

where

$$t_{0j} = jT_0 + t_0, \tag{60}$$

Where t_{0j} is the entrance time of bunch j into the wiggler at $z = 0$. The function $n(\mathbf{r},t)$ is periodic in time, with a period of $T_0 = 2\pi/\omega_0$, so may be represented by the Fourier series

$$n(\mathbf{r},t) = \sum_{n=-\infty}^{\infty} \tilde{n}_n(\mathbf{r}) e^{-in\omega_0 t}, \tag{61}$$

where the n harmonic coefficient of the density $\tilde{n}_n(\mathbf{r})$ is given by

$$\tilde{n}_n(\mathbf{r}) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} n(\mathbf{r},t) e^{in\omega_0 t} dt. \tag{62}$$

Setting Eq. (59) in Eq. (62) results in

$$\tilde{n}_n(\mathbf{r}) = \frac{N_b \omega_0}{2\pi v_z}\delta(\mathbf{r}_\perp) e^{in\omega_0(z/v_z + t_0)}, \tag{63}$$

We assume that all bunches are identical, namely $I(z=0,t) = -eN_b \sum_{j=-\infty}^{\infty} \delta(t - jT_0 - t_0)$ (so that $|\tilde{M}_b| = 1$). This current contains an infinite number of harmonics, but we assume that only the fundamental harmonic at ω_0 is interacting synchronously with the wave, so that we keep only $\tilde{n}_1(\mathbf{r})$ and $\tilde{n}_{-1}(\mathbf{r}) = \tilde{n}_1^*(\mathbf{r})$, as follows:

$$n(\mathbf{r},t) \approx \tilde{n}_1(\mathbf{r}) e^{-i\omega_0 t} + c.c. = Re\{\tilde{n}(\mathbf{r}) e^{-i\omega_0 t}\} \tag{64}$$

so that the phasor definition for the particles density is

$$\tilde{n}(\mathbf{r}) = 2\tilde{n}_1(\mathbf{r}) \tag{65}$$

From the above we may express $\mathbf{v}_\perp$ and derive from it $\tilde{\mathbf{J}}_\perp$:

$$\tilde{\mathbf{J}}_\perp = \frac{Q_b \omega_0 \tilde{\boldsymbol{\beta}}_w^*}{2\pi\beta_z}\delta(\mathbf{r}_\perp) e^{i\int_0^z (\omega_0/v_z(z') - k_w(z'))dz' + i\varphi_{b0}} \tag{66}$$

where $\varphi_{b0} = \omega_0 t_0$ is the entrance phase of the bunched beam. Defining

$$\varphi(z) = \int_0^z \left(\frac{\omega_0}{v_z} - k_w - k_z\right) dz' + \varphi_{b0}, \tag{67}$$

and using Eq. (5) in a phasor context, we obtain

$$\frac{d\tilde{C}_q(z)}{dz} = -\frac{Q_b \omega_0 \tilde{\boldsymbol{\beta}}_w(z) \cdot \tilde{\boldsymbol{\mathcal{E}}}_q^*(0)}{8\pi \mathcal{P}_q \beta_z} e^{i\varphi(z)}, \tag{68}$$

We define the detuning parameter consistent with Eq. (30):

$$\theta(z) \equiv \frac{d\varphi}{dz} = \frac{\omega_0}{v_z(z)} - k_w(z) - k_z. \tag{69}$$

The rate of change of the bunches energy is:

$$\mathbf{E}(\mathbf{r}, t_e(z)) = Re\left[\tilde{C}_q(z)\tilde{\boldsymbol{\mathcal{E}}}(\mathbf{r}_\perp) e^{-i\int_0^z (\omega_0/v_z(z') - k_z)dz' - i\varphi_{b0}}\right], \tag{70}$$

which for a tight bunch ($\mathbf{r}_\perp = 0$) results in

$$mc^2 \frac{d\gamma}{dz} = \frac{1}{2\beta_z}(-e)\eta_p |\tilde{\boldsymbol{\beta}}_w||\tilde{\boldsymbol{\mathcal{E}}}_q(0)||\tilde{C}_q(z)| \cos[\varphi(z) - \varphi_q(z)], \tag{71}$$

where $\varphi(z)$ is defined in (67), $\varphi_q(z)$ is the phase of $\tilde{C}_q(z)$

$$\tilde{C}_q(z) = |\tilde{C}_q(z)| e^{i\varphi_q(z)}. \tag{72}$$

The polarization match factor η_p is defined by

$$\eta_p = \frac{|\tilde{\boldsymbol{\beta}}_w^* \cdot \tilde{\boldsymbol{\mathcal{E}}}_q(0)|}{|\tilde{\boldsymbol{\beta}}_w||\tilde{\boldsymbol{\mathcal{E}}}_q(0)|} \tag{73}$$

It is useful at this point to redefine the phase of the ponderomotive wave as

$$\psi \equiv -[\varphi(z) - \varphi_q(z) - \pi/2], \tag{74}$$

This results in (similarly to [4]):

$$\frac{d\gamma}{dz} = -\frac{e\eta_p}{2\beta_z \gamma mc^2}\overline{a}_w(z)|\tilde{\boldsymbol{\mathcal{E}}}_q(0)||\tilde{C}_q(z)| \sin\psi \tag{75}$$

where we used $|\tilde{\boldsymbol{\beta}}_w| = \overline{a}_w(z)/\gamma$.

Likewise, also the equation for the mode amplitude (68) can be expressed in terms of ψ:

$$\frac{d\tilde{C}_q(z)}{dz} = iBe^{i[\varphi_q(z) - \psi]}, \tag{76}$$

where

$$B = -\frac{Q_b \omega_0 \eta_p \overline{a}_w(z)|\tilde{\boldsymbol{\mathcal{E}}}_q(0)|}{8\pi \mathcal{P}_q \beta_z \gamma}, \tag{77}$$

is a real positive parameter since $Q_b = -eN_b < 0$. We consider now the case of a uniform wiggler, and therefore γ_r (see Eq. 49) is independent of z, so that

$$\frac{d\theta}{dz} = -\frac{k_0}{\beta_{zr}^3 \gamma_{zr}^2 \gamma_r}\frac{d\delta\gamma}{dz} = -\frac{k_0}{\beta_{zr}^3 \gamma_{zr}^2 \gamma_r}\frac{d\gamma}{dz}. \tag{78}$$

The total power of the electron beam can be expressed as

$$P_{el} = \frac{1}{T_0} N_b mc^2(\gamma - 1) \simeq \frac{1}{T_0} N_b mc^2 \gamma, \tag{79}$$

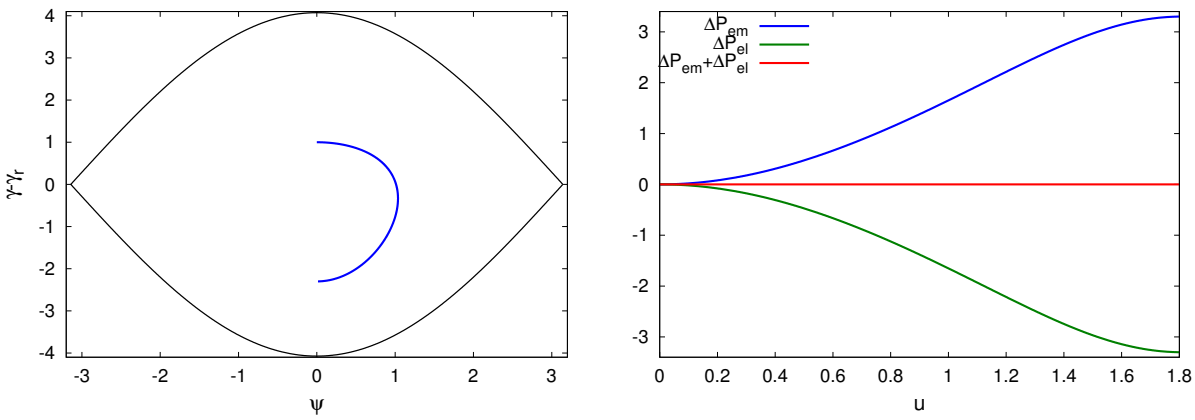

FIG. 4: Panel (a) shows the phase-space diagram $\psi - \theta$, where the black line shows the separatrix at the end of the trajectory. Panel (b) shows the radiation power change, the electron beam power change, and their sum which keeps 0.

We summarize here the equations to be solved:

$$\frac{d|\tilde{C}_q|}{dz} = B \sin\psi, \tag{80}$$

$$\frac{d\varphi_q}{dz} = \frac{B}{|\tilde{C}_q|} \cos\psi, \tag{81}$$

$$\frac{d\theta}{dz} = K_s^2(z) \sin\psi, \tag{82}$$

$$\frac{d\psi}{dz} = -\theta + \frac{B}{|\tilde{C}_q|} \cos\psi \tag{83}$$

where

$$K_s^2(z) = \frac{k_0 e \eta_p}{2\beta_{zr}^4 \gamma_{zr}^2 \gamma_r^2 mc^2} \overline{a}_w |\tilde{\boldsymbol{\mathcal{E}}}_q(0)||\tilde{C}_q(z)| \tag{84}$$

is the synchrotron oscillation wavenumber. In Figure 4 we show the case of maximum energy extraction from a given input radiation wave. The input wave is characterized by $\tilde{C}_q(0) = 1$. For longer runs, we get synchrotron oscillations, so that the beam and the electromagnetic wave oscillated between minimum and maximum power.

It should be pointed out that beyond preliminary design consideration, the tight bunching model of the bunched beam dynamics is only marginally fitting for describing the dynamics of a tapered wiggler FEL, where the beam energy spread, bunching amplitude and phase control are limited. It is, however, a quite good model for describing the dynamics in optical frequency interaction experiments, such as RUBICON and NOCIBUR, where quite tight bunching and bunch phase control are achievable [11, 12].

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Acknowledgments

This research was supported in part by a grant from the United States-Israel Binational Science Foundation(BSF), Jerusalem, ISRAEL

This research was supported in part by the Deutsch-Israelische Projektkooperation (DIP)